PAPER - 9

COMPLEX ANALYSIS - I

Objectives

This course provides

(i) a modern treatment of concepts and techniques of complex function theory

(ii) To gain knowledge about the complex number system, the complex function and complex integration.

UNIT-I: Complex numbers and Elementary functions

Complex Number system, complex numbers –Algebraic properties-Point at Infinity Stereographic Projection-Function of a complex variable-Mappings-Elementary Functions- The Logarithmic function- Branches of log Z.
Sections 1-10, 21-30.

UNIT-II: Analytic functions

Definitions of Limits -Continuity-Derivatives and Differentiation formula-Cauchy-Riemann equations-Cauchy-Riemann equations in polar form-properties of Analytic functions-Necessary and sufficient conditions for Analytic functions-problems. Sections 11-19.

UNIT-III: Conformal Mappings

Harmonic functions-Determination of Harmonic conjugate and Analytic functions-conformal mapping-Isogonal mapping-Further properties and examples-transformations of Harmonic functions.
Sections 20, 76-80.

UNIT-IV Mapping by Elementary transformations

The transformations $w=z+d$, $w=1/z$, $w=z^2$, $\overline{w}=\sqrt{z}$, $w = e^z$, $w=\sin z$ Bilinear Transformation andspecial Bilinear Transformation problems.
Sections 31-36, 38-39

UNIT-V: Integrals

Contours - Line Integrals _ Cauchy-Goursat's Theorem (without proof) Cauchy's Integral Formula - Derivatives of Analytic Functions - problems. Sections 43-46, 50-52.

Recommended Text

R.V.Churchill and J.W.Brown, (1984) *Complex Variables and Applications*. McGraw Hill International Book Co., Singapore. (Third Edition)

Reference Books:

1. P. Duraipandian and LaxmiDuraipandian (1976) Complex Analysis: Emerald Publishers, Chennai
2. S. Ponnusamy. (2000) Foundations of Complex Analysis, Narosa Publishing House, New Delhi
3. Murray R. Spiegel. (2005) Theory and Problems of Complex Variable. Tata-Mcgraw Hill Edition, New Delhi.

UNIT - I

Complex Number

A Complex Number z can be represented as $z = x+iy$, where x, y are real numbers and which has an ordered pair $z = (x, y)$

Properties :-

(i) sum of two complex number is also a complex number

(ie) $z_1 = x_1+iy_1$, $z_2 = x_2+iy_2$

$\therefore z_1+z_2 = (x_1+x_2)+i(y_1+y_2)$

Here z_1, z_2 is a complex number $\Rightarrow z_1+z_2$ is also a complex number.

(ii) product of two complex number is also a complex number.

(iii) The conjugate of complex number $z = x+iy$ is given by $\bar{z} = x-iy$ (Reflection)

Geometric Interpretation of a Complex Number :-

Any Complex number represent a directed line segment

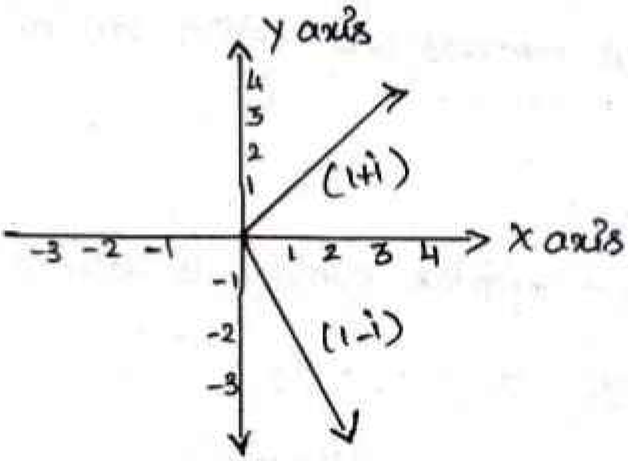
(i.e) Let $z = 1+i$ Then $\bar{z} = 1-i$

we know that $\bar{z}$ is reflection of z

Here $z = (1,1)$ and $\bar{z} = (1,-1)$

Geometrically xy plane is called the complex plane (or) z plane.

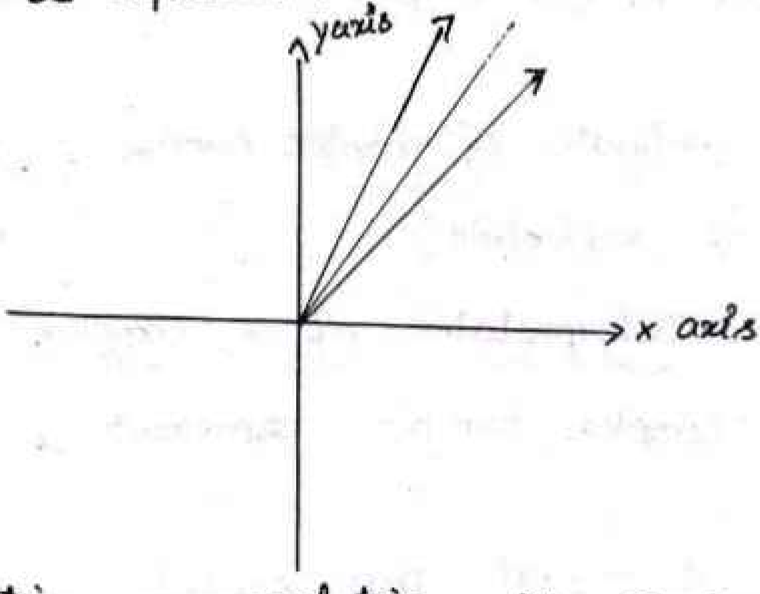
(ie) The x axis is called real axis and Y axis is called imaginary axis and the number z is called the directed line segment or vector representation.



(i) Geometric representation on $z_1 + z_2$

Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ Then $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

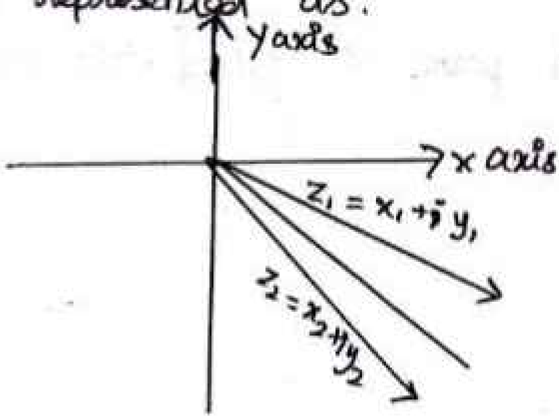
which can be represented as.



(ii) Geometric representation on $z_1 - z_2$

Let $z_1 = x_1 + iy$, $z_2 = x_2 + iy_2$ Then $z_2 - z_1 = (x_2 - x_2) + i(y_2 - y_1)$

which can be represented as.



Definition :-

Modulus or absolute value :-

Modulus or absolute value of a complex number is defined as,

$$|z| = \sqrt{x^2+y^2}$$

Algebraic properties :-

(i) Commutative Law :-

$$z_1+z_2 = z_2+z_1$$

$$z_1 z_2 = z_2 z_1$$

(ii) Associative Law :-

$$(z_1+z_2)+z_3 = z_1+(z_2+z_3)$$

$$z_1(z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$$

(iii) Distributive Law :-

$$z_1 \cdot (z_2+z_3) = z_1 z_2 + z_1 z_3$$

Note :-

Combination of two distributive function remains the same.

(ii) $(z_1+z_2) \cdot z_3 = (z_1+z_3) \cdot z_2$

(iv) Identity Law :-

The additive identity is $0+i0$ and multiplicative identity is $1+i0$ which caries over the entire complex plane

Note :-

Identity is unique

(V) Inverse Law :-

The additive inverse is $z+(-z)=0$ and the multiplicative inverse is $z \cdot \frac{1}{z} = 1$ (or) $zz^{-1} = 1$.

Properties of conjugates :-

(i) $\bar{\bar{z}} = z$

(ii) $|\bar{z}| = |z|$

Proof :-

Let $z = x+iy$ Then $|z| = \sqrt{x^2+y^2} \rightarrow ①$

Let $\bar{z} = x-iy$ Then $|\bar{z}| = \sqrt{x^2+y^2} \rightarrow ②$

① = ②

$\Rightarrow |z| = |\bar{z}|$

(iii) $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$

Proof :-

Let $z_1 = x_1+iy_1$, $\bar{z}_1 = x_1-iy_1$

Let $z_2 = x_2+iy_2$, $\bar{z}_2 = x_2-iy_2$

$z_1+z_2 = (x_1+iy_1)+(x_2+iy_2)$

$= (x_1+x_2)+i(y_1+y_2) \rightarrow$

$\overline{z_1+z_2} = (x_1+x_2)-i(y_1+y_2) \rightarrow ①$

$\bar{z}_1+\bar{z}_2 = (x_1-iy_1)+(x_2-iy_2)$

$= (x_1+x_2)-i(y_1+y_2) \rightarrow ②$

① & ②

$\Rightarrow \overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$

(iv) $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$

Proof :-

Let $z_1 = x_1+iy_1$, $\bar{z}_1 = x_1-iy_1$

$z_2 = x_2+iy_2$, $\bar{z}_2 = x_2-iy_2$

$z_1z_2 = (x_1+iy_1)(x_2+iy_2)$

$= x_1x_2+ix_1y_2+ix_2y_1-y_1y_2$

$\overline{z_1z_2} = (x_1x_2-y_1y_2)-i(x_1y_2+x_2y_1) \rightarrow ①$

$\bar{z}_1 \cdot \bar{z}_2 = (x_1 - iy_1)(x_2 - iy_2)$

$= x_1x_2 - ix_1y_2 - ix_2y_1 + y_1y_2$

$= (x_1x_2 - y_1y_2) - i(x_1y_2 + x_2y_1) \rightarrow ②$

① & ②

$\Rightarrow \overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$

(V) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, \quad z_2 \neq 0$

Proof :-

Let $z_1 = x_1 + iy_1$, $\bar{z}_1 = x_1 - iy_2$

Let $z_2 = x_2 + iy_2$, $\bar{z}_2 = x_2 - iy_2$

$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \times \frac{x_2 - iy_2}{x_2 - iy_2}$

$= \frac{x_1x_2 - ix_1y_2 + iy_1x_2 + y_1y_2}{x_2^2 + y_2^2}$

$= \frac{(x_1x_2 + y_1y_2) + i(y_1x_2 - y_2x_1)}{x_2^2 + y_2^2}$

$= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{y_1x_2 - y_2x_1}{x_2^2 + y_2^2}$

$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} - i\frac{y_1x_2 - y_2x_1}{x_2^2 + y_2^2} \rightarrow ①$

$\frac{\bar{z}_1}{\bar{z}_2} = \frac{x_1 - iy_1}{x_2 - iy_2} \times \frac{x_2 + iy_2}{x_2 + iy_2}$

$= \frac{x_1x_2 + ix_1y_2 - iy_1x_2 + y_1y_2}{x_2^2 + y_2^2}$

$= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{x_1y_2 - y_1x_2}{x_2^2 + y_2^2}$

$$= \frac{x_1x_2 + y_1y_2}{x_1^2+y_1^2} - i\,\frac{y_1x_2 - y_2x_2}{x_2^2+y_2^2} \longrightarrow ②$$

① & ②

$$\Rightarrow \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$$

(vi) $z+\bar{z} = 2R(z)$

Proof:

Let $z = x+iy$, $\bar{z} = x-iy$

$z+\bar{z} = x+iy+x-iy$

$= 2x$

$z+\bar{z} = 2x$

$z+\bar{z} = 2R(z)$

(vii) $z-\bar{z} = 2i\,I(z)$

Proof :-

Let $z = x+iy$, $\bar{z} = x-iy$

$z-\bar{z} = x+iy-x+iy$

$= 2iy$

$z-\bar{z} = 2i\,I(z)$

$z\bar{z} = |z|^2$

Proof:-

Let $z = x+iy$, $\bar{z} = x-iy$

$|z| = \sqrt{x^2+y^2}$, $|z|^2 = x^2+y^2$

$z\bar{z} = (x+iy)(x-iy)$

$= x^2+y^2$

$z\bar{z} = |z|^2$

(8) Properties of moduli :-

(i) $|z_1 z_2| = |z_1||z_2|$

(ii) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $|z_2| \neq 0$

(8) (iii) Triangle property

$$|z_1+z_2| \leq |z_1|+|z_2|$$

(iv) $\big||z_1|-|z_2|\big| \leq |z_1+z_2|$

complex number in polar form :-

Let $x = r\cos\theta$

$y = r\sin\theta$

$\Rightarrow \tan\theta = y/x$

$\theta = \tan^{-1}(y/x)$

and we have $z = r(\cos\theta + i\sin\theta)$

which can be written as

$z = re^{i\theta}$, where ,

θ is the argument of z and the positive number of r is called the length of the vector z.

(ie). $r = |z| \Rightarrow r = \sqrt{x^2+y^2}$

Regions in a Complex plane

ε - neighbourhood

Let z_0 be a given point in a Complex plane, then the ϵ-neighbourhood of the point z_0 is given by $|z-z_0|<\epsilon$ where ϵ is arbitrary.

(ie) . It consist of all points z lying inside the circle with Centre at z_0 and radius ε.

Interior point of S:

A point zo is said to be a interior point of the set S, when there is same neighbourhood of zo that contains only pts. of S.

A pt. zo is an exterior pt. when there exists a nbd. which contains no point of S. zo is a boundary point of the set S when its nbd pt. at in S.

The totality of all boundary points is called boundary of S.

Open set:

A set is open if its contains no boundary points, also each of its pts. is an interior pt.

Closed set:

A set is closed if its contains all its boundary pts.

Closure of S[$\bar{S}$]:

The closure S[$\bar{S}$] of a set is a closed set consisting of all points in S and not in S.

Sets - neigther open nor closed:

There are certain sets which are neigther open nor closed called disc $0 < |z| \leq 1$. is

Connected set:

An open set is said to be connected if each pair of pt. z_1 & z_2 in it can be joined by a polygonal path consisting of finite no. of line segment.

An open set which is not connected is called disc connected set. An open connected set is called a domain.

A domain together with some (or) all of its boundary pts. is called a region.

Bounded :

A set S is bounded if every pt. of S lies inside some ϵ neighbourhood contained in S, otherwise the set is Unbounded.

Accumulation pt.

A pt. z_0 is said to be accumulation pt. of a set S, if each nbd. of z_0 contains atleast one pt. of S distinct from z_0.

Note: If a set S is closed then its contains each of its accumulation points.

Finite Complex plane :

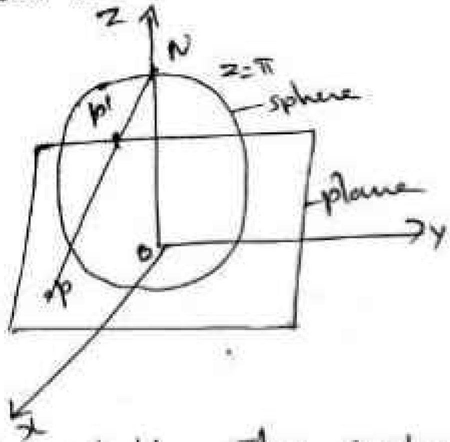
All complex number except infinite are said to be finite complex number and the complex plane containing only the finite complex number is called finite complex plane.

Extended Complex plane:

The complex number plane containing all complex no. together with infinite is called extended complex plane.

(X.) **Stereographic Projection :**

Consider the unit sphere $x^2+y^2+z^2=1$ and the extended complex plane $z=\pi$ to established a pt. to pt. correspondence between the pt. of sphere and the plane.



Consider a pt. p on the plane π. Let $N(0,0,1)$ denote the north pole of the sphere, join PN. Let p' be the pt. of intersection of the line joining p & N. The surface of the sphere $p \to p'$ similarly each pt. of the plane corresponds to pt. of sphere in S.

This one to one Correspondents is called stereographic Projection.

Let $x^2+y^2+z^2=1$ —— (1)

$\therefore$ the pts. $N(0,0,1)$, $P'(X,Y,Z)$, $P(x,y,0)$ are Collinear.

We know that the eqn. of the plane containing the pt.

$$\frac{X-x_1}{l} = \frac{Y-y_1}{m} = \frac{Z-z_1}{n}$$

$$\frac{X}{x} = \frac{Y}{y} = \frac{Z-1}{-1} \quad \text{—— (2)}$$

$$\frac{X}{x} = \frac{Y}{y} = \frac{Z-1}{-1} = r$$

$$X = xr,\ Y = yr,\ \frac{Z-1}{-1} = r \Rightarrow Z = -r+1$$

Sub. in (1).

$$x^2r^2 + y^2r^2 + (-r+1)^2 = 1$$

$$r^2(x^2+y^2+1) = 2r$$

$$r = \frac{2}{x^2+y^2+1}$$

$$X = \frac{2x}{x^2+y^2+1},\ Y = \frac{2y}{x^2+y^2+1},\ Z = \frac{-2}{x^2+y^2+1}+1 = \frac{x^2+y^2-1}{x^2+y^2+1}$$

We have

$$z = x+iy,\ \bar{z} = x-iy$$

$$z+\bar{z} = 2x,\ z-\bar{z} = 2iy$$

$$z\bar{z} = x^2+y^2,\ X = \frac{z+\bar{z}}{z\bar{z}+1} //,\ Y = -\frac{z-\bar{z}}{z\bar{z}+1} = \frac{-i(z-\bar{z})}{z\bar{z}+1}$$

$$= \frac{i(\bar{z}-z)}{z\bar{z}+1} //$$

$$Z = \frac{z\bar{z}-1}{z\bar{z}+1} //$$

Functions of Complex Variable:

Let S be a set of complex number a function f defined on S assigns a rule to each z in S, a complex number w. It is denoted by $w=f(z)$. The set S is called the domain of the function f.

(X) Eg:

W = U+iV is the value of the function of z=x+iy,
If $f(z) = z^2$ separate the function into real and imaginary parts.

Solution:

$$W = U+iV$$
$$f(z) = z^2, \quad W = f(z)$$
$$U+iV = z^2$$
$$U+iV = (x+iy)^2 = x^2-y^2+2ixy$$
$$R.P = x^2-y^2, \quad I.P = 2xy.$$

Polynomial function:

The function $p(z) = a_0+a_1z+a_2z^2+\dots+a_nz^n$, where $a_n \neq 0$, is polynomial function of degree n. The polynomial $\frac{P(z)}{Q(z)}$, $Q(z) \neq 0$ is called as rational function.

Mappings:

The function f is displayed by indicating the pairs z = (x,y), W = (u,v) (ie) we draw the w-plane and z-plane separately. The function f is called a mapping (or) a transformation.

The image of the point z is the point $w = f(z)$.
The set of all pts. of images is called the image of the given set.
The image of entire domain is called range.

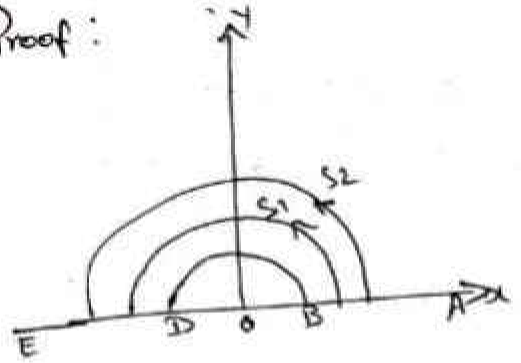
Terms such as translation, rotation and reflection are used to study the Geometric properties of certain mappings.

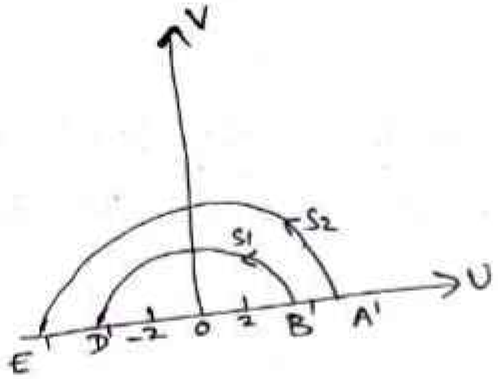
com Eg: S.T under the transformation $W = z+\frac{1}{z}$

(X) (X) Remark (i). The image of domain consisting of the points in the upper half plane y>0, which are exterior to the circle |z|=1 is the entire upper half plane v>0

(ii). The image of boundary of domain in the z-plane is U-axis.

Proof :





$W = z + \frac{1}{z}$ —①

The image of the point $z = x$ $(x>1)$ on the x-axis and the pt. $z=1$ is according to equ.① $W = z + \frac{1}{z}$, the pts. $W = x + \frac{1}{x}$ this image pt. represents a real number in the u plane.

∵ the derivative of the function $x + \frac{1}{x}$ is always positive when $x>1$, the value of W increases as x increases.

Hence as the point $z = x$ $(x \geq 1)$ moves to the right along the x-axis, the image moves to the right along the y-axis.

Starting at the pt. $W = 2$.

Illy if $z = -x$ the image be the $-u$ axis, starting at the pt. $W = -2$. Let $z = re^{i\theta}$

$$\Rightarrow w = re^{i\theta} + \frac{1}{r} e^{-i\theta} \quad \text{—②}$$

The image of the typiccal pt $z = re^{i\theta}$ $(0 \leq \theta \leq \pi)$ on the upper half of the circle $|z|=1$ is given by $x=1 \Rightarrow$ equ.② $w = e^{i\theta} e^{-i\theta} = 2\cos\theta$.

As the value of θ increases from $\theta = 0$ to π, the point $z = e^{i\theta}$ traverse the upper half of the circle in the counter clockwise direction and its image $w = 2\cos\theta$ moves to the left along the U-axis from $W = 2$ to -2.

By equ. (2) the image of the pt. $z = r_1 e^{i\theta}$ on the upper half S_1 of a circle $|z| = r_1$ where $r_1 > 1$

$$\Rightarrow w = r_1 e^{i\theta} + \frac{1}{r_1} e^{-i\theta}$$

$$= r_1 (\cos\theta + i\sin\theta) + \frac{1}{r_1}(\cos\theta - i\sin\theta) -$$

$$r_1 \cos\theta + i r \sin\theta + \frac{1}{r_1}\cos\theta - \frac{1}{r_1}\sin\theta$$

$$= \cos\theta\left(r_1 + \frac{1}{r_1}\right) + i\sin\theta\left(r_1 - \frac{1}{r_1}\right)$$

$$= a\cos\theta + i\, b\sin\theta.$$

Where $a = r_1 + \frac{1}{r_1}$, $b = r_1 - \frac{1}{r_1}$ and we can write $w = u + iv$.

where $u = a\cos\theta$ and $v = b\sin\theta$ ——— (3)

from (3), $u = a\cos\theta$, $v = b\sin\theta$

$\cos\theta = u/a$ $\sin\theta = \frac{v}{b}$

Squaring and adding, we have

$$\cos^2\theta + \sin^2\theta = \frac{u^2}{a^2} + \frac{v^2}{b^2}$$

$$1 = \frac{u^2}{a^2} + \frac{v^2}{b^2} \quad \text{——— (4)}$$ which is an ellipse in the w-plane. When a radius r_2 to larger that r_1 is taken the upper half S_2 of the circle $|z| = r_2$ and its image S_2' are both larger. Infact as the semi circles increases in size their images which are upper half of ellipse with foci at $w = \pm 2$ out the entire upper half plane $v > 0$.

Elementary Functions.

(i) Exponential Functions:

If a function f of the complex variable $z = x+iy$ is to reduce to

(a). $f(x + i0) = e^x$, when z is real, $\forall$ real number x.

(b). f is entire and $f'(z) = f(z)$ $\forall z$

(c). $e^z = e^x(\cos y + i\sin y)$.

Properties:

(i). $e^z = e^{x+iy} = e^x . e^{iy}$

(ii). $(\exp z_1)(\exp z_2) = \exp(z_1+z_2)$

If $e^{z_1} . e^{z_2} = e^{x_1+iy_1} + e^{x_2+iy_2}$

$= e^{x_1+iy_1+x_2+iy_2}$

$= e^{(x_1+x_2)+i(y_1+y_2)}$

$= e^{z_1+z_2}$

(iii). $e^{z+2\pi i} = e^z . e^{2\pi i} = e^z$ $\quad [\because e^{2\pi i} = 1]$

(iv). $\dfrac{e^{z_1}}{e^{z_2}} = e(z_1 - z_2)$

(v). $(e^z)^n = e^{nz}$

(vi). $e^0 = 1$, $\dfrac{1}{e^z} = e^{-z}$

(vii) $|e^z| = e^x$ and $\arg(e^z) = y + 2n\pi$

viii). $e^{i\pi/4} = 1$ & $e^{2n\pi i} = 1$.

Logarithmic function and its branches:

The log function of a non-zero complex variable $z = re^{i\theta}$ is defined by $z = re^{i\theta}$

$\Rightarrow \log z = \log r + i\theta$

$= \log|z| + i(\arg z)$ $\quad$ [since $|z| = r$ and $\theta = \arg(z)$]

The principle values of θ is given by $\theta = \varphi + 2n\pi$

$(n = 0, \pm 1, \pm 2 \cdots)$.

$\log z = \log|z| + i(\varphi + 2n\pi)$

(ie) $\log z$ is multiple valued, those values have the same real part and there imaginary part differ. by integral multiples of 2π.

Properties

(i) $e^{\log z} = z$

(ii) $|e^z| = e^x$ & $\arg(e^z) = y + 2n\pi$

(iii). $\log(e^z) = z + 2n\pi i \quad (n = 0, \pm 1, \pm 2 \ldots)$

(iv). $\log(z_1 z_2) = \log z_1 + \log z_2$

(v). $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

(vi). $\log|z_1 z_2| = \log|z_1| + \log|z_2|$

(vii). $\log|z_1 z_2| + i\arg(z_1 z_2) = (\log|z_1| + i\arg z_1) + (\log|z_2| + i\arg z_2)$

(viii). $\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2$

(ix). $e^{x\log z} = z^n$

(x). $e^{\frac{1}{n}\log z} = z^{\frac{1}{n}}$

(Q) PT $\log z$ is Continuous everywhere and analytic everywhere.

Proof: Let $u = \log r$ and $v = \theta$

$\frac{\partial u}{\partial r} = u_r = \frac{1}{r}$, $v_r = 0$

$\frac{\partial u}{\partial \theta} = u_\theta = 0$, $v_\theta = 1$

Continuous everywhere in that domain and satisfy polar form.

$x = r\cos\theta$ $\qquad y = r\sin\theta$

$\frac{\partial x}{\partial r} = \cos\theta$ $\qquad \frac{\partial y}{\partial r} = \sin\theta$

$\frac{\partial x}{\partial \theta} = -r\sin\theta$ $\qquad \frac{\partial y}{\partial \theta} = r\cos\theta$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y}(r\cos\theta)$$

$$= r\left(-\frac{\partial v}{\partial x}\sin\theta + \frac{\partial v}{\partial y}\cos\theta\right)$$

Using C-R equations.

$u_x = v_y$, $v_x = -u_y$

$$\Rightarrow \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{\partial u}{\partial y}\sin\theta + \frac{\partial u}{\partial x}\cos\theta = \frac{\partial u}{\partial r}$$

$u_r = \frac{1}{r}v_\theta$, $v_r = -\frac{1}{r}v_\theta$

We conclude that $\log z$ is analytic in the domain $r>0, -\pi<\theta<\pi$

$\because \log z = \log r + i\theta$ & $z = re^{i\theta}$

$$r\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = r\left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}\right) + i\left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial r}\right)$$

$$= r\left[\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right) + i\,\frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta)\right]$$

$$= r\cos\theta\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right] + r\sin\theta\left[\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right]$$

$$= x\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + iy\left(\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}\right)$$

$$= x f'(z) + iy f'(z)$$

$$= (x+iy) f'(z) = z f'(z)$$

$$\frac{r}{z}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = f'(z)$$

$$\frac{r}{re^{i\theta}} = \frac{1}{e^{i\theta}} = \frac{1}{z}$$

$$\frac{d}{dz}(\log z) = \frac{1}{z} \quad (|z| > 0,\ -\pi < \arg < \pi)$$

$\Rightarrow$ it is continuous everywhere, hence the function is analytic everywhere.

Branch of $\log z$.

A Branch of a multiple valued function f is any single valued function F which is analytic in some domain at each point z of which the value $F(z)$ is one of the values of $f(z)$. The function $\log z$ when restricted to the domain $r>0, -\pi<\theta<\pi$ is the branch of log function which is called principle branch.

A singular point common to all branch cut for a multiple valued function is called branch point. The origin is the branch point of ~~log function~~ log fn.

The ray $\varphi = \pi$ is the branch cut for the principle branch ; a branch cut being a P of a line (or) a curve consisting of singular points in order to define a branch of multiple valued function.

Commutative property of Addition :-

Addition of two complex number is commutative a complex number then $z_1 = (a,b)$ and $z_2 = (c,d)$ are two complex number.

For commutative property of addition.

$$z_1 + z_2 = z_2 + z_1$$

Proof :-

Let $z_1 = a+ib \quad , z_2 = c+id$

$$z_1 + z_2 = a+ib+c+id$$
$$= (a+c)+i(b+d) \longrightarrow ①$$

$$z_2 + z_1 = (c+id) + (a+ib)$$
$$= (c+a)+i(d+b) \longrightarrow ②$$

From ① = ②

$$\Rightarrow z_1 + z_2 = z_2 + z_2$$

Associative property of Addition :-

Addition of three complex number is associative

If $z_1 = (a,b)$, $z_2 = (c,d)$ and $z_3 = (e,f)$ Then

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$$

Proof :-

$$z_1 + (z_2 + z_3) = (a+ib) + [(c+id) + (e+if)]$$
$$= (a+ib) + [(c+e) + i(d+f)]$$
$$= (a+c+e) + i(b+d+f) \longrightarrow ①$$

$$(z_1 + z_2) + z_3 = [(a+ib) + (c+id)] + (c+if)$$
$$= [(a+c) + i(b+d)] + (c+if)$$
$$= (a+c+e) + i(b+d+f) \longrightarrow ②$$

From ① = ②

$\Rightarrow z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

Additive Identity :-

The Complex number $(0,0)$ is the additive identity for addition of complex number. $(0,0)$ is called as zero complex number.

Proof :-

If $z = (a,b)$ and $0 = (0,0)$ Then

$z + 0 = 0 + z = z$

Existence of Inverse :-

The negative complex number is the additive inverse of a complex number.

If $z = (a,b)$ Then the inverse of this complex number is

$-z = (-a,-b)$

$z + (-z) = (-z) + z$

Proof :-

$z + (-z) = a + ib + (-a - ib)$

$= a + ib - a - ib$

$= 0 \longrightarrow ①$

$(-z) + z = -a - ib + a + ib$

$= 0 \longrightarrow ②$

From ① = ② $\Rightarrow z + (-z) = (-z) + z$

Commutative property of multiplication :-

Multiplication of two complex number is commutative.

If z represents a complex number then $z_1 = (a,b)$ and $z_2 = (c,d)$ are two complex numbers. For commutative property

$z_1 \cdot z_2 = z_2 \cdot z_1$

Proof :-

$z_1 = a+ib \quad , z_2 = c+id$

$z_1 . z_2 = (a+ib).(c+id)$

$= ac + iad + ibc - bd$

$z_1 . z_2 = (ac - bd) + i(ad + bc) \longrightarrow ①$

$z_2 . z_1 = (c+id).(a+ib)$

$= ca + icb + ida - db$

$= (ca - db) + i(cb + da) \longrightarrow ②$

From ① = ② $\Rightarrow z_1 z_2 = z_2 . z_1$

Associative property multiplication :-

Multiplication of three complex number is Associative

$z_1 . (z_2 . z_3) - (z_1 . z_2) . z_3$

Proof :-

If $z_1 = (a,b) \quad , \quad z_2 = (c,d) \quad , \quad z_3 = (e,f)$

Then ,

$z_1 . (z_2 , z_3) = (a+ib) . [(c+id).(e+if)]$

$= (a+ib).(ce - df + i(cf + de)]$

$= (a+ib). [ce - df + i(cf + de)]$

$= ace - adf + iacf + iade + ibce - ibdf$
$- bcf - bde$

$= [ace - adf - bcf - bde] +$
$i[acf + ade + bce - bdf] \longrightarrow ①$

$(z_1 . z_2) . z_3 = [(a+ib).(c+id)]. (e+if)$

$= [ac + iad + ibc - bd]. (e+if)$

$= ace + iacf + iade - adf + ibce$
$- bcf - bde - ibdf$

$= [ace - adf - bcf - bde] + i$
$[acf + ade + bce - bdf] \longrightarrow ②$

From ① = ②

$\Rightarrow z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$

Distributive Law's :-

If z_1, z_2, z_3 are any three complex numbers which are not equal to zero, Then.

$$z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$$

Proof :-

Let $z_1 = a+ib$, $z_2 = c+id$ and $z_3 = e+if$ then

$$z_1 \cdot (z_2 + z_3) = (a+ib) \cdot [(c+id) + (e+if)]$$
$$= (a+ib) \cdot [(c+e) + i(d+f)]$$
$$= a(c+e) + ia(d+f) + ib(c+e) - b(d+f)$$
$$= ac + ae + iad + iaf + ibc + ibe - bd - bf$$
$$= (ac + ae - bd - bf) + i(ad + af + be + be) \rightarrow ①$$

$$z_1 \cdot z_2 = (a+ib) \cdot (c+id)$$
$$= ac + iad + ibc - bd$$
$$= (ac - bd) + i(ad + bc)$$

$$z_1 \cdot z_3 = (a+ib) \cdot (e+if)$$
$$= ae + iaf + ibe - bf$$
$$= (ae - bf) + i(af + be)$$

$$z_1 \cdot z_2 + z_1 \cdot z_3 = (ac - bd + ae - bf) + i(ad + bc + af + be) \rightarrow ②$$

From ① = ②

$$z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$$

Identity Element of multiplication :

The complex number $(1, 0)$ is the identity or unity for multiplication.

If $z = a+ib$ Then $z.(1,0) = (a+ib).(1+i0)$

$= a+ib+0$

$= a+ib$

$= z \longrightarrow (1)$

$(1,0).z = (1+i0).(a+ib)$

$= a+ib$

$= z \longrightarrow (2)$

From $(1) = (2)$

Inverse element of Multiplication :

If $z = a+ib \neq (0,0)$. then it has the multiplicative inverse.

$$\frac{1}{z} = \frac{1}{a+ib}$$

$$z.\frac{1}{z} = a+ib \cdot \frac{1}{a+ib} = 1$$

Triangular property :-

The modulus of sum of two complex number is always less than or equal to sum of their moduli.

(ie) $|z_1+z_2| \leq |z_1| + |z_2|$

Proof:-

Let z_1 and z_2 be two complex numbers.

we know that $|z_1+z_2|^2 = (z_1+z_2)(\overline{z_1+z_2})$

$[\because |z|^2 = z\bar{z}]$

$= (z_1+z_2)(\bar{z}_1+\bar{z}_2)$

$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2$

$= z_1\bar{z}_1 + z_2\bar{z}_2 + z_1\bar{z}_2 + \overline{z_1\bar{z}_2}$

$= |z_1|^2 + |z_2|^2 + 2Re(z_1\bar{z}_2)$ $\quad \because Re(z) \leq |z|$

$\leq |z_1|^2 + |z_2|^2 + 2|z_1\bar{z}_2|$ $\quad \because |\bar{z}| = |z|$

$= |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$

$= [|z_1| + |z_2|]^2$

Taking positive square root we get,

$$|z_1+z_2| \le |z_1|+|z_2|$$

Properties:

(i) $e^z = e^{x+iy} = e^x \cdot e^{iy}$

Proof :-

$z = x+iy$

$e^z = e^{x+iy}$

$e^z = e^x \cdot e^{iy}$

Hence the proof

ii) $(\exp z_1)(\exp z_2) = \exp(z_1+z_2)$

Proof :-

$z_1 = x_1+iy_1, \quad z_2 = x_2+iy_2$

$$(\exp z_1)(\exp z_2) = [\exp(x_1+iy_1)][\exp(x_2+iy_2)]$$
$$= \exp[(x_1+iy_1)+(x_2+iy_2)]$$
$$= \exp(z_1+z_2)$$

$$(\exp z_1)(\exp z_2) = \exp(z_1+z_2)$$

Hence the proof

iii) $e^{z+i2\pi} = e^z \cdot e^{2\pi i} = e^z \quad [\because e^{2\pi i} = 1]$

Proof :-

$z = x+iy$

$$e^{z+i2\pi} = e^{(x+iy)+i2\pi}$$
$$= e^{x+iy} \cdot e^{2\pi i}$$
$$= e^{x+iy} \cdot 1$$
$$e^{z+i2\pi} = e^z$$

Hence the proof

(iv) $\dfrac{e^{z_1}}{e^{z_2}} = e^{(z_1-z_2)}$

Proof :-

$$\frac{e^{z_1}}{e^{z_2}} = \frac{e^{x_1+iy_1}}{e^{x_2+iy_2}}$$

$$= e^{x_1+iy_1} \, e^{-(x_2+iy_2)}$$

$$= e^{x_1+iy_1-(x_2+iy_2)}$$

$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$$

Hence the proof

(v) $e^0 = 1$; $\dfrac{1}{e^z} = e^{-z}$

Proof :-

$$\frac{1}{e^z} = \frac{1}{e^{x+iy}}$$

$$= e^{-(x+iy)}$$

$$\frac{1}{e^z} = e^{-z}$$

(vi) $(e^z)^n = e^{nz}$

Proof :-

$$(e^z)^n = (e^{x+iy})^n \qquad (\because z = x+iy)$$

$$= e^{n(x+iy)}$$

$$= e^{nz}$$

$$(e^z)^n = e^{nz}$$

Hence the proof.

Problems :-

(1) find the $\sqrt{-7+24i}$

soln:-

$\sqrt{-7+24i} = z$

$\sqrt{-7+24i} = x+iy$

squaring on both sides

$-7+24i = (x+iy)^2$

$-7+24i = x^2-y^2+2ixy$

Equating real & imaginary part

$-7 = x^2-y^2 \rightarrow ①$

$24i = 2ixy$

$24 = 2xy \rightarrow ②$

$② \Rightarrow x = 12/y$

sub in eqn ①

$-7 = \left(\frac{12}{y}\right)^2 - y^2$

$-7 = \frac{144}{y^2} - y^2$

$\frac{144-y^4}{y^2} = -7$

$144-y^4 = -7y^2$

$y^4-7y^2-144=0$

put $t - y^2$

$t^2-7t-144=0$

-144

$-16 \quad 9$

-7

$(t-16)(t+9)=0$

$t=16 \quad t=-9 = t=9i^2$

$y^2=16 \quad y^2=9i^2$

$y=\pm 4 \quad y=\pm 3i$

when $y=\pm 4$

$x = \pm 12/4 = \pm 3$

when $y = \pm 3i$

$$x = \pm \frac{12}{3i} = \pm 4i$$

$\therefore z = x+iy$

$= 3 \pm 4i$

(2) $\sqrt{5+6i}$

Soln :-

$\sqrt{8+6i} = z$

$\sqrt{8+6i} = x+iy$

squaring on both sides

$8+6i = (x+iy)^2$

$8+6i = x^2-y^2+2ixy$

Equating real & imaginary part

$8 = x^2-y^2 \longrightarrow ①$

$6 = 2xy \longrightarrow ②$

$x^2+y^2 = \sqrt{(x^2-y^2)^2+(2xy)^2}$

$= \sqrt{8^2+6^2} = \sqrt{64+36}$

$= \sqrt{100}$

$x^2+y^2 = 10 \longrightarrow ③$

Adding ① + ③

$2x^2 = 18$

$x^2 = 18/2 = 9$

sub in eqn ②

$2xy = 6$ $\quad$ $2x-3y = 6$

$2 \times 3y = 6$ $\quad$ $-6y = 6$

$6y = 6$ $\quad$ $y = -1$

$y = 1$

$y = \pm 1$

$z = x+iy$

$z = 3 \pm i$

(3) Find the $\sqrt{9+7i}$

soln:-

$\sqrt{9+7i} = z$

$\sqrt{9+7i} = x+iy$

$9+7i = (x+iy)^2$

$9+7i = x^2+y^2+2ixy$

Equating real & imaginary part

$x^2-y^2 = 9 \rightarrow ①$

$2xy = 7 \rightarrow ②$

$x^2+y^2 = \sqrt{(x^2-y^2)^2+(2xy)^2}$

$= \sqrt{9^2+7^2} = \sqrt{81+49}$

$= \sqrt{130}$

$x^2+y^2 = \sqrt{130} \rightarrow ③$

$① + ③ \Rightarrow$ $2x^2 = 9+\sqrt{130}$

$x^2 = \frac{9}{2} + \frac{\sqrt{130}}{2}$

$x = \pm\sqrt{\frac{9}{2} + \frac{\sqrt{130}}{2}}$

sub in eqn ②

$2\sqrt{\frac{9}{2}+\frac{\sqrt{130}}{2}}\; y = 7$

$y = \pm \frac{7}{2} \frac{1}{\sqrt{\frac{9}{2} + \frac{\sqrt{130}}{2}}}$

$z = x+iy$

$= \pm\sqrt{\frac{9}{2}+\frac{\sqrt{130}}{2}} \pm i \frac{7}{2\sqrt{\frac{9}{2}+\frac{\sqrt{130}}{2}}}$

H.W (4).

$\sqrt{1-\sqrt{3}i}$

soln :-

$$\sqrt{1-\sqrt{3}i} = z$$
$$= x+iy$$
$$1-\sqrt{3}i = x^2-y^2+2iyx$$

Equating real and imaginary part

$$x^2-y^2 = 1 \longrightarrow ①$$
$$2xy = -\sqrt{3} \rightarrow ②$$
$$x^2+y^2 = \sqrt{(x^2-y^2)^2+(2xy)^2}$$
$$= \sqrt{1^2+3} = \sqrt{4} = 2$$
$$x^2+y^2 = 2 \longrightarrow ③$$

① + ③

$$2x^2 = 3$$
$$x^2 = 3/2$$
$$x = \pm\sqrt{3/2}$$

sub in eqn ②

$$2\sqrt{\frac{3}{2}}\,y = -\sqrt{3}$$
$$y = \frac{-\sqrt{3}}{2\sqrt{3/2}}$$
$$= -\frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{\sqrt{3}}$$
$$y = -\frac{1}{\sqrt{2}}$$
$$z = x+iy$$
$$z = \sqrt{\frac{3}{2}} - i\frac{1}{\sqrt{2}}$$

(ii) $\sqrt{2i}$

soln:-

$\sqrt{2i} = z$

$\sqrt{2i} = x+iy$

squaring on both sides

$0+2i = x^2-y^2+2ixy$

Equating real & imaginary part

$x^2-y^2 = 0 \rightarrow (1)$

$2xy = 2 \rightarrow (2)$

$x^2+y^2 = \sqrt{(x^2-y^2)^2+(2xy)^2}$

$= \sqrt{0^2+2^2}$

$= \sqrt{4} = 2$

$x^2+y^2 = 2 \longrightarrow (3)$

(1) + (3)

$2x^2 = 2$

$x^2 = 1$

$x = \pm 1$

sub in eqn (2)

$2 \times 1 y = 2$

$2y = 2$

$y = \pm 1$

$z = x+iy$

$z = 1 \pm i$

(6) $\sqrt{-8-6i}$

soln :-

$\sqrt{-8-6i} = z$

$\sqrt{-8-6i} = x+iy$

squaring on both sides

$-8-6i = x^2-y^2+2ixy$

Equating real and imaginary part

$x^2-y^2 = -8 \rightarrow (1)$

$2xy = -6 \rightarrow (2)$

$x^2+y^2 = \sqrt{(x^2-y^2)^2+(2xy)^2}$

$= \sqrt{(-8)^2+(-6)^2} = \sqrt{64+36} = \sqrt{100} = 10$

$x^2+y^2 = 10 \rightarrow (3)$

(1)+(3)

$$\begin{array}{r} x^2-y^2 = -8 \\ x^2+y^2 = 10 \\ \hline 2x^2 = 2 \end{array}$$

$x^2 = 1 \Rightarrow x = \pm 1$

sub in eqn (2)

$2(1)y = -6$

$2y = -6$

$y = -6/2$

$y = -3$

$z = x+iy$

$= 1 \pm 3i$

(7) Find all the values of $(\sqrt{3}+i)^{2/3}$

soln:-

Let $z = re^{i\theta}$ then the values of

$\sqrt{3}+i = re^{i\theta}$

$\sqrt{3+i} = r(\cos\theta + i\sin\theta) \rightarrow (1)$

Equating real & Imaginary part

$\sqrt{3} = r\cos\theta$ $\quad$ $\sqrt{3} = 2\cos\theta$ $\quad$ $1 = 2\sin\theta$

$1 = r\sin\theta$ $\quad$ $\frac{\sqrt{3}}{2} = \cos\theta$ $\quad$ $\frac{1}{2} = \sin\theta$

$\theta = \pi/6$ $\quad$ $\theta = \pi/6$

$r = \sqrt{a^2+b^2}$

$= \sqrt{(\sqrt{3})^2+1^2} = \sqrt{3+1} = \sqrt{4} = 2$

$r = 2$

sub in (1)

$\sqrt{3+i} = 2(\cos\theta + i\sin\theta)$

$(\sqrt{3+i})^{2/3} = 2^{2/3}(\cos\theta + i\sin\theta)^{2/3}$ & by De moivre's theorem

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$= 2^{2/3}(\cos\pi/6 + i\sin\pi/6)^{2/3}$

$= 2^{2/3}\left(\cos\pi/6 \times \frac{2}{3} + i\sin\frac{\pi}{6}\times\frac{2}{3}\right)$

$= 2^{2/3}\left(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right)$

$= 2^{2/3}\left[\cos\left(2k\pi + \frac{\pi}{9}\right) + i\sin\left(2k\pi + \frac{\pi}{9}\right)\right]$

$k = 0, 1, 2$

$k = 0$

$= 2^{2/3}\left[\cos\frac{\pi}{9} * i\sin\frac{\pi}{9}\right]$

$k = 1$

$= 2^{2/3}\left[\cos\frac{19\pi}{9} + i\sin\frac{19\pi}{9}\right]$

$k = 2$

$= 2^{2/3}\left[\cos\frac{37\pi}{9} + i\sin\frac{37\pi}{9}\right]$

(8) Find all the values of $8^{1/6}$

Soln :-

Let $z = re^{i\theta}$ then the values of

$8^{1/6} = re^{i\theta}$

$8 + i0 = r(\cos\theta + i\sin\theta) \rightarrow (1)$

$8 + i0 = r(\cos\theta + i\sin\theta)$

Equating real & imaginary part

$8 = r\cos\theta$

$0 = \sin\theta$

$r = \sqrt{a^2+b^2}$

$= \sqrt{8^2+0^2} = \sqrt{64} = 8$

$r = 8$

$8 = 8\cos\theta$

$1 = \cos\theta$

$\theta = 0$

$0 = \sin\theta$

$\theta = 0$

Sub in eqn (1)

$8 + i0 = 8(\cos\theta + i\sin\theta)$

$(8)^{1/6} = 8^{1/6}(\cos\theta + i\sin\theta)^{1/6}$ & by

De'moivre's theorem.

$= 8^{1/6}(\cos 0 + i\sin 0)^{1/6}$

$= 8^{1/6}(\cos 0 + i\sin 0)$

$= 8^{1/6}[\cos(2k\pi + 0) + i\sin(2k\pi + 0)]$

$k = 0$ $\qquad\qquad k = 0, 1, 2, 3, 4, 5$

$= 8^{1/6}[\cos 0 + i\sin 0]$

$k = 1$

$= 8^{1/6}[\cos 2\pi + i\sin 2\pi]$

$k=2 \quad = 8^{1/6}\left[\cos 4\pi + i\sin 4\pi\right]$

$k=3 \quad = 8^{1/6}\left(\cos 6\pi + i\sin 6\pi\right)$

$k=4 \quad = 8^{1/6}\left[\cos 8\pi + i\sin 8\pi\right]$

$k=5 \quad = 8^{1/6}\left[\cos 10\pi + i\sin 10\pi\right]$

(9) Find the value of $e^{z} = 1+\sqrt{3}\,i$

soln:-

$e^{z} = 1+\sqrt{3}i$

$e^{x+iy} = 1+\sqrt{3}i$

$e^{x} = e^{iy} - 1+\sqrt{3}i$

Equating real and imaginary part

$e^{x} = 1 \qquad e^{y} - \sqrt{3}$

$x = 0$

principle value is

$y = 2n\pi + \sqrt{3}$

(or)

$y = 2n\pi \pm \pi/3$

where $\theta = \tan^{-1}(y/x)$

$\theta = \tan^{-1}(\sqrt{3}/1)$

$\theta = \pi/3$

$e^{z} = 2$

soln:-

$e^{z} = 2+i0$

$e^{x+iy} = 2+i0$

$e^{x}\cdot e^{iy} = 2+i0$

$e^{z} = 2 \quad = e^{z} = 2\cdot e^{i\pi/4}$

$e^{x+iy} = 2\cdot e^{i\pi/4}$

$$e^{x} \cdot e^{iy} = 2 \cdot e^{i\pi/4}$$

Equating real & imaginary part

$$e^{x} = 2 \qquad e^{y} = e^{\pi/4}$$

$$x = \log 2 \qquad y = 2n\pi \pm \pi/4$$

$$e^{4z} = i$$

soln :-

$$e^{4z} = 0+i$$

$$e^{4(x+iy)} = 0+i$$

$$e^{4x+i4y} = 0+i$$

$$e^{4x} \cdot e^{i4y} = 0+i$$

Equating real & imaginary part

$$e^{4x} = 0 \qquad e^{4y} = 1$$

$$4x^{4} = \log 0 \qquad 4y = 2n\pi \pm \pi/4$$

$$x^{4} = 1 \qquad y = \frac{1}{4}(2n\pi \pm \pi/4)$$

$$x = 1^{1/4} \qquad y = \frac{n\pi}{2} \pm \frac{\pi}{16}$$

H.W

(10) $e^{2z-1} = 1$

soln:-

$$e^{2(x+iy)-1} = 1.1$$

$$e^{2x-1+2iy} = 1 . e^{i\pi/4}$$

$$e^{2x-1} \cdot e^{i2y} = 1 . e^{i\pi/4}$$

Equating ~~real~~ and imaginary part

$e^{2x-1} = 1$ $\qquad$ $e^{2y} = e^{\pi/4}$

$x^2-1 = 0$ $\qquad$ $2y = 2n\pi \pm \pi/4$

$x^2 = 1$ $\qquad$ $y = \frac{1}{2}(2n\pi \pm \pi/4)$

$x = \pm 1$ $\qquad$ $y = n\pi \pm \frac{\pi}{8}$

(i) $(2i)^{1/5}$

soln:-

Let $z = re^{i\theta}$ Then the values of

$0+2i = re^{i\theta}$

$0+2i = r(\cos\theta + i\sin\theta) \rightarrow ①$

Equating real and imaginary part

$0 = r\cos\theta$

$2 = \sin\theta$

$r = \sqrt{a^2+b^2} = \sqrt{0+2^2} = \sqrt{4}$

$r = 2$

$0 = 2\cos\theta$

$\cos\theta = 0$

$\theta = \pi/2$

$2 = 2\sin\theta$

$\sin\theta = 1$

$\theta = \pi/2$

sub in eqn ①

$0+2i = 2(\cos\theta + i\sin\theta)$

$(2i)^{1/5} = 2^{1/5}(\cos\theta + i\sin\theta)^{1/5}$ &

by De' moivre's Theorem

$$= 2^{1/5} (\cos \pi/2 + i \sin \pi/2)^{1/5}$$

$$= 2^{1/5} \left(\cos \frac{\pi}{10} + i \sin \frac{\pi}{10} \right)$$

$$= 2^{1/5} \left[\cos \left(2k\pi + \frac{\pi}{10} \right) + i \sin \left(2k\pi + \frac{\pi}{10} \right) \right]$$

$k = 0, 1, 2, 3, 4$

$$k = 0 \Rightarrow 2^{1/5} \left[\cos \frac{\pi}{10} + i \sin \frac{\pi}{10} \right]$$

$$k = 1 \Rightarrow 2^{1/5} \left[\cos \frac{21\pi}{10} + i \sin \frac{21\pi}{10} \right]$$

$$k = 2 \Rightarrow 2^{1/5} \left[\cos \frac{41\pi}{10} + i \sin \frac{41\pi}{10} \right]$$

$$k = 3 \Rightarrow 2^{1/5} \left[\cos \frac{61\pi}{10} + i \sin \frac{61\pi}{10} \right]$$

$$k = 4 \Rightarrow 2^{1/5} \left[\cos 81\pi/10 + i \sin \frac{81\pi}{10} \right]$$

ii) $(-1+i)^{2/5}$

sol:-

Let $z = re^{i\theta}$ Then the values of

$$-1+i = r(\cos\theta + i \sin\theta) \rightarrow ①$$

Equating the real & imaginary part

$$-1 = r\cos\theta$$

$$1 = r\sin\theta$$

$$r = \sqrt{a^2+b^2} = \sqrt{1^2+1^2} = \sqrt{2}$$

$$-1 = \sqrt{2}\cos\theta$$

$$\cos\theta = \frac{1}{\sqrt{2}}$$

$$\theta = \pi/4$$

$$1 = \sqrt{2}\sin\theta$$

$\sin\theta = \frac{1}{\sqrt{2}}$

$\theta = \pi/4$

sub in eqn ①

$-1+i = \sqrt{2}\,(\cos\theta + i\sin\theta)$

$(-1+i)^{2/5} = (\sqrt{2})^{2/5}\,(\cos\theta + i\sin\theta)^{2/5}$ &

by De' moive's theorem

$= (\sqrt{2})^{2/5}\,(\cos\pi/4 + i\sin\pi/4)^{2/5}$

$= (2)^{1/2 \times 2/5}\left(\cos\frac{3\pi}{30} + i\sin\frac{3\pi}{30}\right)$

$= (2)^{1/5}\left[\cos\left(2k\pi + \frac{2\pi}{30}\right) + i\sin\left(2k\pi + \frac{2\pi}{30}\right)\right]$

$= (2)^{1/5}\left[\cos\left(2k\pi + \frac{\pi}{10}\right) + i\sin\left(2k\pi + \frac{\pi}{10}\right)\right]$

$k = 0, 1, 2, 3, 4$

$k=0 \Rightarrow (2)^{1/5}\left[\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right]$

$k=1 \Rightarrow (2)^{1/5}\left[\cos\frac{21\pi}{10} + i\sin\frac{21\pi}{10}\right]$

$k=2 \Rightarrow (2)^{1/5}\left[\cos\frac{41\pi}{10} + i\sin\frac{41\pi}{10}\right]$

$k=3 \Rightarrow (2)^{1/5}\left[\cos\frac{61\pi}{10} + i\sin\frac{61\pi}{10}\right]$

$k=4 \Rightarrow (2)^{1/5}\left[\cos\frac{81\pi}{10} + i\sin\frac{81\pi}{10}\right]$

(iii) $(-8-8\sqrt{3}i)^{1/4}$

soln:-

Let $z = re^{i\theta}$ Then the values of

$-8-8\sqrt{3}i = r(\cos\theta + i\sin\theta) \rightarrow ①$

Equating real and imaginary part

$$-8 = r\cos\theta$$

$$-8\sqrt{3} = r\sin\theta$$

$$r = \sqrt{a^2+b^2} = \sqrt{(-8)^2+(-8\sqrt{3})^2}$$

$$= \sqrt{64+64\times 3}$$

$$= \sqrt{64+192} = \sqrt{256}$$

$$r = 16$$

$$-8 = 16\cos\theta \qquad -8\sqrt{3} = 16\sin\theta$$

$$\cos\theta = -8/16 = -1/2 \qquad \sin\theta = \frac{-8\sqrt{3}}{16} = -\sqrt{3}/2$$

$$\theta = \pi/3 \qquad \theta = \pi/3$$

sub in eqn ①

$$-8-8\sqrt{3}i = 16(\cos\theta + i\sin\theta)$$

$$(-8-8\sqrt{3}i)^{1/4} = (16)^{1/4}(\cos\theta + i\sin\theta)^{1/4} \quad \&$$

By De'marive's theorem

$$= (16)^{1/4}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{1/4} //$$

UNIT - II

Analytic Function :

Defn :

Function of a Complex Variable :

Generaly a Complex number is denoted by $z = x+iy$.

Consider another Complex number $w = u+iv$

Let $w = z^2$

$$u+iv = (x+iy)^2$$
$$= (x^2-y^2) + i(2xy)$$

Comparing real and imaginary part $u = x^2-y^2$, $v = 2xy$

Here to each point z of a region R there corresponds atleast one point w of a region R. we say that w is a function of z and we write $w = f(z)$.

Defn :

single valued function and Multiple valued function.

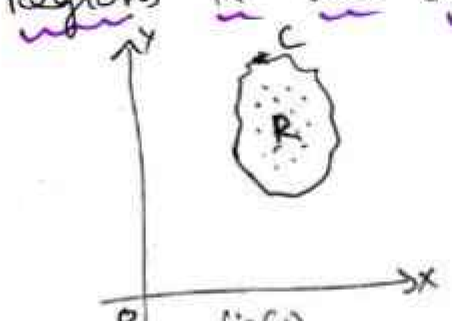
For each value of z, if a unique value exists, then w is called a single valued function of z.

Eg : $w = z^2$ is a single valued function of z for each value of z, if more then one value of w exists, then w is called a multipled valued function.

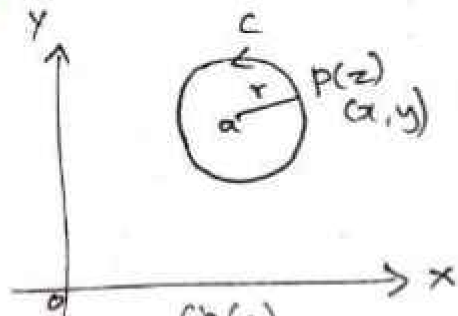
Eg : $w = \sqrt{z}$ is two valued function of z.

since w assume two values corresponding to each value of z, except $z=0$ (when $z=0$ then w is called a branch point).

Regions in the Complex plane

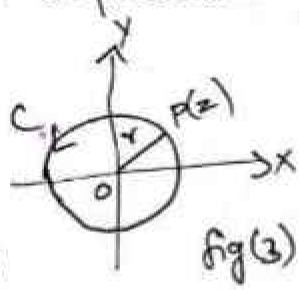


fig(1)



fig(2)

Let 'z' and 'a' be any two points in the Complex plane of the region R (fig 1). Then the distance between the two points is $|z-a|$. Consider the circle 'c' with radius 'r' and with centre 'a' then the eqn. of the circle 'c' is represented by $|z-a|=r$ fig(2)



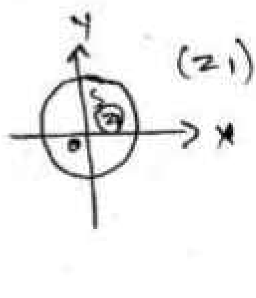
fig(3)

If we take the centre as the origin 'o' with readius 'r' then the circle is denoted by $|z|=r$ (fig(3))

The Inequality $|z-a|<r$ holds good for any point inside 'c' (ie) It represent the interior of the circle excluding its circumference, the interior of the circle including the circumference is represented by $|z-a|\leq r$

The inequality $|z-a|>r$ represented the exterior of the circle 'c'.

$\therefore$ The equation $|z|=1$ represent the circle with centre origin and of radius 1. This circle is called unit circle.



Defn: Limit Point

Let s be a set of Complex number

then the Complex number z_1 is called

Limit pt. of Set S if every nbd. of z_1 contains atleast one element of S. otherthan z_1, z_1 may (or) may not be belong to the set.

Defn: Limit of a function.

Let f be a function define in a set S and let z_0 be a limit point of S. Here z_0 may (or) may not belong to S then f(z) is said to have a complex number l as its limit at $z = z_0$ for any $\epsilon > 0$, $\exists$ a real no / $\delta = \delta(\epsilon)$ such that $|f(z) - l| < \epsilon$ $\forall$ $z \in S$, other than z_0, $|z - z_0| < \delta$

Eg:
$$f(z) = \begin{cases} z^2, & z \neq i \\ 0, & z = i \end{cases} \quad \text{where } z \to i$$

$f(z) = i^2 = -1$, $\lim\limits_{z \to i} f(z) = -1$

Continuous function:

Let f(z) be define and have a single valued in a nbd at $z = z_0$ the function f(z) is said to be a continuous at $z = z_0$ if (i) $f(z_0)$ must exist (ie) f(z) is defined at z_0.

(ii) $\lim\limits_{z \to z_0} f(z) = l$ must exist.

(iii) $\lim\limits_{z \to z_0} f(z) = f\left[\lim\limits_{z \to z_0} (z)\right]$

If the above condition satisfy then f(z) is said to be continuous function.

Eg: If $f(z) = z^2$ $\forall$ z then $\lim\limits_{z \to i} f(z) = f(i) = i^2 = -1$

Here f(z) is continuous at $z = i$

Derivative:

If $w = f(z)$ is a single valued in some region R of the argand plane the derivative of f(z) is defined as

$$\frac{df}{dz} = f'(z) = \underset{\Delta z \to 0}{Lt} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

Provided the limit exists independent of the manner in which $\Delta z \to 0$, In such case we say that $f(z)$ is differentiable at z.

Ex:-

find the derivative of $f(z) = z^3 - 2z$ at $z = -1$

$$f'(z) \frac{df}{dz} = 3z^2 - 2$$

$$f'(-1) = \left(\frac{df}{dz}\right)_{z=-1} = 3(-1) - 2$$

$$= 3 - 2 = 1$$

$\therefore$ The derivative of $f'(z)$ exists.

Analytic Function :-

If $f(z)$ is a function define in a Region R Then it is said to be analytic function if the following condition are satisfy.

(1) To every pt $z \in R$, There corresponds a definite value of $f(z)$.

(2) $f(z)$ is continuous of z in the region R.

(3) At every pt $z \in R$ $f(z)$ has a uniquely determine derivative $f'(z)$.

Cauchy - Riemann Condition (Cartesian)

A necessary and sufficient condition that $w = f(z) = u(x,y) + iv(x,y)$ be analytic in a region R. is that the cauchy Rieman equation.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; ; \; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$ are satisfy in R.

where the above four partical derivative an continuous in R.

Necessary condition :

In order that $f(z)$ to be analytic

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)] - [u(x,y) + iv(x,y)]}{\Delta x + i\Delta y} \longrightarrow ①$$

must exists independent of the manner in which Δz (Δx & Δy) approaches zero

There are two possiblities

case (i)

$\Delta y = 0 \quad \Delta x \to 0$

$\therefore$ from equation ①

$$f'(z) = \lim_{\Delta x \to 0} \frac{[u(x+\Delta x, y) + iv(x+\Delta x, y)] - [u(x,y) + iv(x,y)]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[u(x+\Delta x, y) - u(x,y)]}{\Delta x} + i \frac{[v(x+\Delta x, y) - v(x,y)]}{\Delta x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \rightarrow (2)$$, Provided the partial derivative must exists.

Case (ii)

$$\Delta x = 0 \ , \ \Delta y \to 0$$

Then from eqn (1)

$$f'(z) = \lim_{\Delta y \to 0} \frac{[u(x, y+\Delta y) + i v(x, y+\Delta y)] - [u(x,y) + i v(x,y)]}{i \Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{[u(x, y+\Delta y) - u(x,y)] + i[v(x, y+\Delta y - v(x,y)]}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{1}{i} \left[\frac{u(x, y+\Delta y - u(x,y)}{\Delta y}\right] + \left[\frac{v(x, y+\Delta y - v(x,y)}{\Delta y}\right]$$

$$f'(z) = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \longrightarrow (3)$$

f(z) cannot possiblely be analytic unless there two limit are identical.

$\therefore$ f(z) to be analytic eqn (2) & (3) must be equal

$$f'(z) = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \qquad \left[\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial x}\right]$$

$$= \frac{\partial v}{\partial y} + i \frac{\partial v}{\partial x} \qquad \left[\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\right]$$

II

sufficient condition :-

If $w = f(z) = u+iv$ is analytic at the pt z, then it is necessary that the four partical derivative $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ should exists.

Hence suppose that the above four partial derivative are continuous,

$$\therefore \Delta u = u(x+\Delta x, y+\Delta y) - u(x, y)$$

$$= [u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y)] + [u(x, y+\Delta y) - u(x,y)]$$

$$= \left(\frac{\partial u}{\partial x} + \epsilon_1\right)\Delta x + \left(\frac{\partial u}{\partial y} + m_1\right)\Delta y$$

$$\Delta u = \left(\frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\cdot\Delta y\right) + (\epsilon_1 \Delta x + m_1 \Delta y) \rightarrow ④$$

where ϵ_1 and $\eta_1 \rightarrow 0$ as $\Delta x \rightarrow 0$, $\Delta y \rightarrow 0$

lly

$$\Delta v = \left(\frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y\right) + (\epsilon_2 \Delta x + \eta_2 \Delta y) \rightarrow ⑤$$

where ϵ_2 and $\eta_2 \rightarrow a$ as $\Delta x \rightarrow 0$, $\Delta y \rightarrow 0$

But $w = u+iv$

$$\Delta w = \left(\frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y\right) + (\epsilon_1 \Delta x + \eta_1 \Delta y) + i\left(\frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y\right) + i(\epsilon_2 \Delta x + \eta_2 \Delta y)$$

$$\Delta w = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta y + \epsilon\Delta x + \eta\Delta y \rightarrow ⑥$$

where $\epsilon = \epsilon_1 + i\epsilon_2$

$\eta = \eta_1 + i\eta_2$

Hence $\epsilon \rightarrow 0$, $\eta \rightarrow 0$ as $\Delta x \rightarrow 0$ and $\Delta y \rightarrow 0$

By Cauchy Riemann equations,

eqns ⑥ can be written as,

$$\Delta w = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\Delta y + \epsilon\Delta x + \eta\Delta y$$

$$\Delta w = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\Delta x + i\Delta y) + (\epsilon\Delta x + \eta\Delta y) \rightarrow ⑦$$

Dividing ⑦ by $\Delta z = \Delta x + i\Delta y$ and take limit $\Delta z \to 0$

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \right) \left(\frac{\Delta x + i\Delta y}{\Delta x + i\Delta y} \right) + 0$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial x}$$

$$\frac{dw}{dz} = f'(z) \quad \text{from ②}$$

$\therefore$ The derivative $f'(z)$ exists and is unique

(ie) $f(z)$ is analytic in the region R

This complete the proof.

Note :-

If $f(z)$ is analytic then,

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad -⑫$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad -⑬$$

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \left[\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial x} \right]$$

$$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \qquad \left[\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \right]$$

Harmonic function (or) Laplace equation.

Statement:

A necessary condition that $f(z) = u + iv$ be analytic in the region R, is that in the region R u and v satisfies Laplace equation.

Proof:

If $f(z)$ is analytic in the region R then by Cauchy Reimann eqn.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \rightarrow (1) \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \rightarrow (2)$$

are satisfied in the region R

Aussming that u and v are continuous having second order partial derivatives.

Diff (1) partially w.r.t x

IIIly partially Diff (2) w.r.t y

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$$

$$-\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial y \partial x} \rightarrow (4)$$

From (3) & (4) we get

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

eqn ⑤ is said to be a harmonic function of u.

$|||^{ly}$ Partially Diff ① w.r.t y and

Partially Diff ② w.r.t x

we get

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \longrightarrow ⑥$$

eqn ⑥ is said to be a harmonic function of v.

Hence the two functions u and v must be a pair of solution of Laplace of equation.

∴ If a function in two variables x and y satisfy the Laplace equation the function is said to be harmonic function.

Defn :-

Conjugate harmonic (or) conjugate function

If $f(z) = u + iv$ is analytic in a regio R then u and v are called conjugate harmonic function (or) conjugte function.

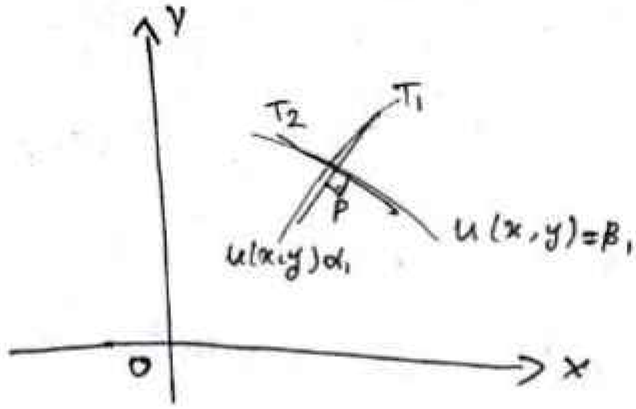
B.W : 3

If $f(z) = u + iv$ is an analytic function then $u(x, y) = \alpha$ and $v(x, y) = \beta$ where α and β are any constant represent two families of curves P.T The families are orthogonal at their pts of intersection.

Proof :-

consider any members of the respective families $u(x, y) = \alpha$ and $v(x, y) = \beta$, where α_1 & β_1 are arbitrary.

Let us take $u(x,y) = \alpha_1$



Differentiating w.r.t x

$$\frac{du}{dx} = 0$$

$$du = 0$$

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$$

(ie) $\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$

Dividing by dx

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0$$

$$\left(\frac{\partial u}{\partial y}\right)\frac{dy}{dx} = -\frac{\partial u}{\partial x}$$

$$\frac{dy}{dx} = -\frac{\partial u/\partial x}{\partial u/\partial y}$$

$$\frac{dy}{dx} = -\left(\frac{\partial u/\partial x}{\partial u/\partial y}\right), (m_1)$$

which represent the slope of the curve $u(x,y) = \alpha_1$

$|||^{ly}$ The slope of the curve $v(x,y) = \beta_1$ is

$$\frac{dy}{dx} = -\left(\frac{\partial v/\partial x}{\partial v/\partial y}\right) (m_2)$$

$\therefore$ The product of the slope

$$(m_1 . m_2) = \frac{(\partial u/\partial x)(\partial v/\partial x)}{(\partial u/\partial y)(\partial v/\partial y)}$$

By C.R equation.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$= \frac{\left(\frac{\partial v}{\partial y}\right)\left(\frac{\partial v}{\partial x}\right)}{\left(-\frac{\partial v}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right)}$$

$$m_1 m_2 = -1$$

Since the product of the slope is -1 the curves intersect at the rt angle at their pt of it

Cor : 1

Determination of conjugate function

If $f(z) = u + iv$ is an analytic function If $u(x, y)$ is given then to determined by $v(x, y)$

Proof :

$$dv = \frac{\partial v}{\partial x} \cdot dx + \frac{\partial v}{\partial y} \cdot dy$$

By C.R equation

$$dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \rightarrow ①$$

Integrating eqn ① on the both sides we get v which is the conjugate function v.

lll^{ly} If v is given u can be determined by the above method.

B.W: 4

Polar form of cauchy Riemann condition :

Statement :

If $f(z)$ is an analytic function which is expressed in polar form $f(z) = u(r, \theta) + iv(r, \theta)$ Then ,

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

Proof :-

The relation between the cartesian and polar

$x = r\cos\theta \quad , \quad y = r\sin\theta$

squaring and adding

$$x^2 + y^2 = r^2 \longrightarrow (1)$$

also $y/x = \tan\theta$

$$\theta = \tan^{-1}(y/x) \longrightarrow (2)$$

Partially differentiating (1) w.r.t x

$$\cancel{2}x = \cancel{2}r\frac{\partial r}{\partial x}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \frac{r\cos\theta}{r} = \cos\theta$$

$$\boxed{\frac{\partial r}{\partial x} = \cos\theta}$$

partially diff (1) w.r.t 'y'

$$\cancel{2}y = \cancel{2}r\frac{\partial r}{\partial y}$$

$$\frac{\partial r}{\partial y} = r\frac{\sin\theta}{r}$$

$$\boxed{\frac{\partial r}{\partial y} = \sin\theta}$$

Partially Diff (2) w.r.t x

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+(y^2/x^2)}\left(-y/x^2\right)$$

$$= \frac{1}{\frac{x^2+y^2}{\cancel{x^2}}}\left(\frac{-y}{\cancel{x^2}}\right) = -\frac{y}{x^2+r^2}$$

$$= -\frac{r \sin\theta}{r}$$

$$\frac{\partial\theta}{\partial x} = -\frac{\sin\theta}{r}$$

partially Diff ② w.r.t y

$$\frac{\partial\theta}{\partial y} = \frac{1}{1+(y^2/x^2)}\left(\frac{1}{x}\right)$$

$$= \frac{1}{\frac{x^2+y^2}{x^{\cancel{2}}}} \frac{1}{\cancel{x}}$$

$$= \frac{x}{x^2+y^2}$$

$$= \frac{r\cos\theta}{r^2}$$

$$\frac{\partial\theta}{\partial y} = \frac{\cos\theta}{r}$$

we known that

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial\theta}\cdot\frac{\partial\theta}{\partial x}$$

$$= \frac{\partial u}{\partial r}\cos\theta + \frac{\partial u}{\partial\theta}\left(-\frac{\sin\theta}{r}\right)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial\theta}\left(\frac{\sin\theta}{r}\right) \longrightarrow ①$$

lll^{ly}

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial\theta}\cdot\frac{\partial\theta}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial\theta}\frac{\cos\theta}{r} \longrightarrow ②$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

$$= \frac{\partial v}{\partial r}\cos\theta + \frac{\partial v}{\partial \theta}\left(-\frac{\sin\theta}{r}\right)$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\cos\theta - \frac{\partial v}{\partial \theta}\frac{\sin\theta}{r} \longrightarrow (3)$$

lll'y

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\cdot\frac{\partial \theta}{\partial y}$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\sin\theta + \frac{\partial v}{\partial \theta}\frac{\cos\theta}{r} \longrightarrow (4)$$

By Cauchy Riemann condition in cartesion

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\left(\frac{\sin\theta}{r}\right) = \frac{\partial v}{\partial r}\sin\theta + \frac{\partial v}{\partial \theta}\frac{\cos\theta}{r} \longrightarrow (5)$$

$$\frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\left(\frac{\cos\theta}{r}\right) = +\frac{\partial v}{\partial \theta}\frac{\sin\theta}{r} - \frac{\partial v}{\partial r}\cos\theta \longrightarrow (6)$$

(5) $\cos\theta$ + (6) $\sin\theta$ gives

$$\frac{\partial u}{\partial r}\cos^2\theta - \frac{\partial u}{\partial \theta}\frac{\cancel{\sin\theta}\cos\theta}{r} = \frac{\partial v}{\partial r}\cancel{\sin\theta}\cos^2\theta + \frac{\partial v}{\partial \theta}\frac{\cos^2\theta\sin\theta}{r}$$

$$\frac{\partial u}{\partial r}\sin^2\theta + \frac{\partial u}{\partial \theta}\frac{\sin\cancel{\cos\theta}}{r} = \frac{\partial v}{\partial \theta}\frac{\sin^2\theta}{r} - \frac{\partial v}{\partial r}\cancel{\sin\theta}\cos\theta$$

Add

$$\frac{\partial u}{\partial r}(\sin^2\theta + \cos^2\theta) = \frac{\partial v}{\partial \theta}\frac{1}{r}(1)$$

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}}$$

⑤ $\sin\theta$ - ⑥ $\cos\theta$ gives

$$\frac{\partial v}{\partial r}\cancel{\cos\theta}\sin\theta - \frac{\partial u}{\partial \theta}\frac{\sin^2\theta}{r} = \frac{\partial v}{\partial r}\sin^2\theta + \frac{\partial v}{\partial \theta}\frac{\sin\cancel{\theta}\cos\theta}{\cancel{r}}$$

$$\frac{\partial v}{\partial r}\cancel{\sin\theta}\cos\theta + \frac{\partial u}{\partial \theta}\frac{\cos^2\theta}{r} = \frac{\partial v}{\partial \theta}\frac{\sin\cancel{\theta}\cos\theta}{\cancel{r}} - \frac{\partial v}{\partial r}\cos^2\theta$$

Sub

$$-\frac{1}{r}\frac{\partial v}{\partial \theta}(1) = \frac{\partial v}{\partial r}(1)$$

$$\boxed{\frac{\partial v}{\partial \theta} = -r\frac{\partial v}{\partial r}}$$

B.W:5

To find $f(z)$ directly using Milne - Thomson method.

In this Method whether u is given and v is given we can find $f(z)$ directly.

Proof :-

$z = x+iy$, $\bar{z} = x-iy$

Adding $z+\bar{z} = 2x$

$\therefore x = \frac{z+\bar{z}}{2}$

subtracting $z-\bar{z} = 2iy$

$\therefore y = \frac{z-\bar{z}}{2i}$

$f(z) = u(x,y) + iv(x,y)$

$= u\left[\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right] + iv\left[\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right]$

The above relation can be regarded as a formal identity. In two independent variable z and $\bar{z}$

$$x = \bar{z}$$

$$f(z) = u(z,0) + iv(z,0)$$

w.k.T $\quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

By C.R equation

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

Let us take $\frac{\partial u}{\partial x} = \phi_1(x,y)$, $\frac{\partial u}{\partial y} = \phi_2(x,y)$

∴ The above equation written as

$$f'(z) = \phi_1(x,y) - i\phi_2(x,y) \quad (x = z, y = 0)$$

$$= \phi_1(z,0) - i\phi_2(z,0)$$

Integrating both sides w.r.t z

$$\boxed{f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\,dz + c} \rightarrow (1)$$

Equation (1) is called Milne's thomson equation

ll^{ly} $\quad$ w.k.T $\quad f'(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$

Let us take $\frac{\partial v}{\partial y} = \psi_1(x,y)$, $\frac{\partial v}{\partial x} = \psi_2(x,y)$

$$\boxed{f(z) = \int [\psi_1(z,0) + i\psi_2(z,0)]\,dz + c} \rightarrow (2)$$

Equation (2) is called Milne's thomson equation.

Problem :-

S.T the function $e^x(\cos y + i\sin y)$ is holomorphic (Analytic) and find its derivative.

Solution :-

$$e^x(\cos y + i\sin y) = e^x \cdot e^{iy} = e^{x+iy} = e^z$$

Let $f(z) = u+iv = e^x(\cos y + i\sin y)$

Comparing real and imaginary part

$u = e^x \cos y \qquad v = e^x \sin y$

$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial v}{\partial x} = e^x \sin y$

$\frac{\partial u}{\partial y} = e^x(-\sin y) \qquad \frac{\partial v}{\partial y} = e^x \cos y$

$= -e^x \sin y$

For $f(z)$ to be holomorphic

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$e^x \cos y = e^x \cos y$$

III^{ly}

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$-e^x \sin y = -e^x \sin y$$

C.R condition satisfy

$\therefore f(z)$ is holomorphic

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= e^x \cos y + i e^x \sin y$$

$$= e^x(\cos y + i\sin y)$$

$= e^{x} e^{iy}$

$= e^{x+iy}$

$f'(z) = e^{z}$

$\boxed{\therefore f'(z) = e^{z}}$

② verify the fallowing function for C-R conditions

(i) $w = az+b$

Sol:

$w = f(z) = az+b$

$u+iv = a(x+iy)+b$

$= ax+b+iay$

Equating real and imaginary part

$u = ax+b \qquad v = ay$

$\frac{\partial u}{\partial x} = a \qquad \frac{\partial v}{\partial x} = 0$

$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial y} = a$

By C.R conditions

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$a = a$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$0 = 0$

$\therefore$ C.R conditions are satisfy

$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$

(ii) $w = z^3 - 2z$

Sol:-

$u+iv = (x+iy)^3 - 2(x+iy)$

$= x^3 + 3x^2 iy - 3xy^3 - iy^3 - 2x - 2iy$

$= x^3 - 3xy^2 - 2x + i(3x^2y - y^3 - 2y)$

Equating real and imaginary part

$u = x^3 - 3xy^2 - 2x$ $\quad$ $v = 3x^2y - y^3 - 2y$

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 - 2$ $\quad$ $\frac{\partial v}{\partial x} = 6xy$

$\frac{\partial u}{\partial y} = -6xy$ $\quad$ $\frac{\partial v}{\partial y} = 3x^2 - 3y^2 - 2$

C.R conditions

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 3x^2 - 3y^2 - 2$

$\therefore$ C.R conditions are satisfy.

(iii) $w = \frac{1}{z}$

Sol:-

$w = u+iv = \frac{1}{z} = \frac{1}{x+iy} \times \frac{x-iy}{x-iy}$

$= \frac{x-iy}{x^2+y^2}$

$u+iv = \frac{x-iy}{x^2+y^2}$

$u = \frac{x}{x^2+y^2}$ $\quad$ $v = \frac{-y}{x^2+y^2}$

$\frac{\partial u}{\partial x} = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2}$ $\quad$ $\frac{\partial v}{\partial x} = -\left[\frac{(x^2+y^2)(0 - y2x)}{(x^2+y^2)^2}\right]$

$= \frac{y^2-x^2}{(x^2+y^2)^2}$ $\quad$ $= 2yx/(x^2+y^2)^2$

$$\frac{\partial u}{\partial y} = \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2} \qquad \frac{\partial v}{\partial y} = -\left[\frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2}\right]$$

$$= -\frac{(x^2-y^2)}{(x^2+y^2)^2} = \frac{(y^2-x^2)}{(x^2+y^2)^2}$$

C.R conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \frac{-2yx}{(x^2+y^2)^2}$$

C.R conditions satisfy

(iii) $w = 2x + ixy^2$

sol:-

$w = u + iv = 2x + ixy^2$

Equating real and imaginary part

$u = 2x \qquad v = xy^2$

$\frac{\partial u}{\partial x} = 2 \qquad \frac{\partial v}{\partial x} = y^2$

$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial y} = 2xy$

C.R condition

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$2 \neq 2xy \qquad 0 = -y^2$$

$\therefore$ C.R conditions are not satisfy

(iv) $w = \log z$

Soln:-

$$w = \log(z) = \log(x+iy)$$

$$u+iv = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}[y/x)$$

$$u = \frac{1}{2}\log(x^2+y^2) \qquad v = \tan^{-1}(y/x)$$

$$\frac{\partial u}{\partial x} = \frac{1}{2}\,\frac{2x}{x^2+y^2} \qquad \frac{\partial v}{\partial x} = \frac{1}{1+(y^2/x^2)}(-y/x^2)$$

$$= \frac{1}{\frac{x^2+y^2}{x^2}}(-y/x^2)$$

$$= \frac{-y}{x^2+y^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{\frac{y^2}{x^2}} \cdot \frac{1}{x}$$

$$= \frac{x}{x^2+y^2}$$

C.R conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \frac{y}{x^2+y^2}$$

$\therefore$ C-R Conditions Satisfy

(20) II

(5) Determine the constant a and b in order that the function $w = x^2+ay^2-2xy+i(bx^2-y^2+2xy)$ should be analytic find the derivative of w.

Sol:-

$$u+iv = w = x^2+ay^2-2xy+i(bx^2-y^2+2xy)$$

$u = x^2+ay^2-2xy$ $\quad$ $v = bx^2-y^2+2xy$

$\frac{\partial u}{\partial x} = 2x-2y$ $\quad$ $\frac{\partial v}{\partial x} = 2bx+2y$

$\frac{\partial u}{\partial y} = 2ay-2x$ $\quad$ $\frac{\partial v}{\partial y} = -2y+2x$

$u_{xx} = 2$ $\quad$ $v_{xx} = 2b$

$u_{yy} = 2a$ $\quad$ $v_{yy} = -2$

given that $f(z)$ is analytic

i.e; by Laplace equation

$$u_{xx}+u_{yy} = 0$$

$$2+2a = 0$$

$$2a = -2$$

$$\boxed{a = -1}$$

By C.R condition

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$2ay-2x = -2bx-2y$$

$$2(-1)y-2x = -2bx-2y$$

$$-2y-2x = -2bx-2y$$

$$-2bx = -2x$$

$$\boxed{b = 1}$$

$$\therefore w = x^2-y^2-2xy+i(x^2-y^2+2xy)$$

we haie to find the derivative of w

$$\frac{dw}{dz} = f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\left[\frac{\partial v}{\partial x} = 2bx + 2y\right]$$

$$\left[= 2(1)x + 2y\right]$$

$$= 2x - 2y + i(2x + 2y)$$

$$f'(z) = 2\left[(x-y) + i(x+y)\right]$$

UNIT - III

Def: Harmonic Function :

If u is a function of x and y

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$ is called the Laplace equation

A function of x and y is said to be a harmonic function if,

1). It has continous first and second order partial derivatives.

2) It satisfies the Laplace equation.

Ex :-

$u = x^3 - 3xy^2$ is a harmonic equation.

1 S.T the function $u(x,y) = e^{-x}(x\sin y - y\cos y)$ is harmonic

Sol:-

$$u(x,y) = e^{-x}(x\sin y - y\cos y)$$

$$u_x = -e^{-x}(x\sin y - y\cos y) + e^{-x}(\sin y)$$

$$u_{xx} = +e^{-x}(x\sin y - y\cos y) - e^{-x}\sin y - e^{-x}\sin y$$

$$= e^{-x}[x\sin y - y\cos y - 2\sin y] \longrightarrow (1)$$

$$u_y = e^{-x}[x\cos y - (\cos y + y(-\sin y))]$$

$$u_{yy} = e^{-x}[-x\sin y - (-\sin y - (\sin y + y\cos y))]$$

$$= e^{-x}[-x\sin y + \sin y + \sin y + y\cos y]$$

$$= e^{-x}[y\cos y - x\sin y + 2\sin y] \longrightarrow (2)$$

$$u_{xx} + u_{yy} = e^{-x}[x\sin y - y\cos y - 2\sin y] + e^{-x}[y\cos y - x\sin y + 2\sin y]$$

$= 0$; u is harmonic

② Find the function $w(f(z))$ S.T $u+iv$ is analytic

(i) $u = x$

$$\frac{\partial u}{\partial x} = 1$$

$$\frac{\partial u}{\partial y} = 0$$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \quad \text{(By C-R condition)}$$

$$dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

$$= 0\, dx + 1\, dy$$

$$\int^{b} dv = dy$$

$$v = y + c$$

$$f(z) = u + iv$$

$$= x + iy + c$$

$$w = z + c$$

[milne - thomson method

$$u_x = 1 = \phi_1(z, 0)$$

$$u_y = 0 = \phi_2(x, y)$$

$$f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\, dz$$

$$= \int (1 - i0)\, dz$$

$$= \int dz$$

$$f(z) = z + c$$]

(ii) $u = \cos x \cos y$

$$\frac{\partial u}{\partial x} = -\sin x \cos y$$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

By C.R

$$= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

$$dv = \sin y \cos x\, dx - \sin x \cos y\, dy$$

$\int^{u}$ both sides

$$v = \sin y \sin x - \sin x \sin y + c$$

$$v = 0 + c$$

$$w = u + iv$$

$$w = \cos x \cos y + c$$

ii) $u = \cos x \cosh y$

$u_x = -\sin x \cosh y$

$u_y = \cos x \sinh y$

By C.R eqn

$u_x = v_y$

$u_y = -v_x$

$v = \int dv = \int \frac{\partial v}{\partial x} dx + \int \frac{\partial v}{\partial y} dy$

$= \int (-\cos x \sinh y) dx + (-\sin x \cosh y) dy$

$= -\sin x \sinh y - \sin x \sinh y + c$

$= -2 \sin x \sinh y + c$

$f(z) = w = u + iv$

$= \cos x \cosh y - i\, 2 \sin x \sinh y + c$

iii) $u = x^3 - 3xy^2 + y + 1$

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$

$\frac{\partial u}{\partial y} = -6xy + 1$

$v = \int dv = \int \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$

$= \int -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$

$= \int (6xy - 1) dx + (3x^2 - 3y^2) dy$

$= 6\frac{x^2}{2} y - x + 3x^2 y - \frac{3y^3}{3}$

$v = 3x^2 y + 3x^2 y - y^3 - x + c$

$w = u + iv = x^3 - 3xy^2 + y + 1 + i(3x^2 y + 3x^2 y - y^3 - x) + c$

$= x^3 - 3xy^2 + y + 1 + i(6x^2 y + y^3 \; x) + c$

③ S.T the function $u = \frac{1}{2} \log (x^2+y^2)$ is harmonic

$$u_x = \frac{1}{\cancel{2}} \frac{1}{(x^2+y^2)} \cancel{2}x \qquad u_y = \frac{1}{\cancel{2}} \frac{1}{(x^2+y^2)} \cancel{2}y$$

$$= \frac{x}{x^2+y^2} \qquad = \frac{y}{x^2+y^2}$$

$$u_{xx} = \frac{(x^2+y^2)1 - x(2x)}{(x^2+y^2)^2} \qquad u_{yy} = \frac{(x^2+y^2) - y2y}{(x^2+y)^2}$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$u_{xx} = \frac{y^2-x^2}{(x^2+y^2)^2} \qquad u_{yy} = -\frac{(y^2-x^2)}{(x^2+y^2)^2}$$

$$u_{xx} + u_{yy} = \frac{y^2-x^2}{(x^2+y^2)^2} - \frac{(y^2-x^2)}{(x^2+y^2)^2}$$

$$= 0$$

$\therefore$ u is harmonic

④ S.T $x^3 - 3xy^2 + 3x + 1$ is harmonic

$$u_x = 3x^2 - 3y^2 + 3 \qquad u_y = -6xy$$

$$u_{xx} = 6x \qquad u_{yy} = -6y$$

$$u_{xx} + u_{yy} = 6x$$

$\therefore$ u is harmonic

⑤ If n is real S.T $r^n [\cos n\theta + i \sin n\theta]$ is analytic possibly when $r=0$

sol:-

$$w = r^n [\cos n\theta + i \sin n\theta]$$

$$u + iv = r^n [\cos n\theta + i \sin n\theta]$$

Equivating real and imaginary part

$u = r^n \cos n\theta \qquad v = r^n \sin n\theta$

$\frac{\partial u}{\partial r} = n r^{n-1} \cos n\theta \; ; \quad \frac{\partial v}{\partial r} = n r^{n-1} \sin n\theta$

$\frac{\partial u}{\partial \theta} = r^n (-\sin n\theta) n \; ; \quad \frac{\partial v}{\partial \theta} = r^n (\cos n\theta) n$

$= -n r^n \sin n\theta$

By C-R equation

$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \quad , \quad \frac{\partial v}{\partial r} = \frac{-1}{r} \frac{\partial u}{\partial \theta}$

$n r^{n-1} \cos n\theta = \frac{1}{r} n r^n \cos n\theta \quad , \quad n r^{n-1} \sin n\theta = \frac{-1}{r} (-n r^n \sin n\theta$

$n r^{n-1} \cos n\theta = n r^{n-1} \cos n\theta \qquad n r^{n-1} \sin n\theta = n r^{n-1} \sin n\theta$

C.R conditions satisfies

The given function is analytic

[Hence the fun exists for all finite values of R in adding zero except when $r = 0$

$\therefore$ It is analytic]

⑥ For what values of z the function w define by the function $e^{-v} (\cos u + i \sin u)$ ceases to be analytic.

Sol:-

w in the fun of z

The fun of u and v. when u and v are fun of x and y.

$\frac{dz}{dw} = \frac{\partial z}{\partial u}$

Let $z = e^{-v} (\cos u + i \sin u)$

$\frac{\partial z}{\partial u} = e^{-v} (-\sin u + i \cos u)$

$$= e^{-v} i [\cos u + i \sin u]$$

$$= i e^{-v} [\cos u + i \sin u]$$

$$\frac{dz}{dw} = iz$$

Hence w ceases to be analytic

when $\frac{dz}{dw} = 0$

$iz = 0$

$i \neq 0, z = 0.$

RESULT :

① S.T

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial^2}{\partial z \partial \bar{z}}$$

Proof :-

$z = x + iy$

$\bar{z} = x - iy$

$z + \bar{z} = 2x$

$\therefore x = \frac{z + \bar{z}}{2}$

$y = \frac{z - \bar{z}}{2i}$

$y = \frac{-i}{2}(z - \bar{z})$

$\frac{\partial x}{\partial z} = 1/2 \qquad \frac{\partial y}{\partial z} = -i/2$

$\frac{\partial x}{\partial \bar{z}} = 1/2 \qquad \frac{\partial y}{\partial \bar{z}} = i/2$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x} \cdot \frac{\partial x}{\partial z} + \frac{\partial}{\partial y} \cdot \frac{\partial y}{\partial z}$$

$$= \frac{\partial}{\partial x} \frac{1}{2} + \frac{\partial}{\partial y} (-i/2)$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\partial/\partial y\right)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}$$

$$= \frac{\partial}{\partial x}\,\tfrac{1}{2} + \partial/\partial y\,(i/2) = \tfrac{1}{2}\left(\partial/\partial x + i\,\partial/\partial y\right)$$

$$\frac{\partial^2}{\partial z\partial \bar{z}} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

$$\therefore \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\partial \bar{z}}$$

(2) If $f(z) = u+iv$ is an analytic fun of z and φ is any function of x and y with differential co-efficient of first two order then P.T.

(i) $\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 = \left[\left(\frac{\partial\varphi}{\partial u}\right)^2 + \left(\frac{\partial\varphi}{\partial v}\right)^2\right]|f'(z)|^2$

(ii) $\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = \left(\frac{\partial^2\varphi}{\partial u^2} + \frac{\partial^2\varphi}{\partial v^2}\right)|f'(z)|^2$

Sol:-

(i) $$\frac{\partial\varphi}{\partial x} = \frac{\partial\varphi}{\partial u}\cdot\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial v}\cdot\frac{\partial v}{\partial x} \longrightarrow (1)$$

$$\frac{\partial\varphi}{\partial y} = \frac{\partial\varphi}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial\varphi}{\partial v}\cdot\frac{\partial v}{\partial y} \longrightarrow$$

By C.R equation

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

$$\therefore \frac{\partial\varphi}{\partial y} = \frac{\partial\varphi}{\partial u}\left(-\frac{\partial v}{\partial x}\right) + \frac{\partial\varphi}{\partial v}\left(\frac{\partial u}{\partial x}\right)$$

$$= -\frac{\partial\varphi}{\partial u}\cdot\frac{\partial v}{\partial x} + \frac{\partial\varphi}{\partial v}\,\frac{\partial u}{\partial x} \longrightarrow (2)$$

$① ^2 + ② ^2 \Rightarrow$

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 = \left(\frac{\partial \psi}{\partial u}\right)^2 \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial v}\right)^2 \left(\frac{\partial v}{\partial x}\right)^2 + 2\,\frac{\partial \psi}{\partial u}\cdot\frac{\partial u}{\partial x}\cdot\frac{\partial \psi}{\partial v}\frac{\partial v}{\partial x}$$

$$+ \left(\frac{\partial \psi}{\partial u}\right)^2 \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial v}\right)^2 \left(\frac{\partial u}{\partial x}\right)^2 - 2\,\frac{\partial \psi}{\partial u}\cdot\frac{\partial v}{\partial x}\cdot\frac{\partial \psi}{\partial v}\cdot\frac{\partial u}{\partial x}$$

$$= \left(\frac{\partial \psi}{\partial u}\right)^2 \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] + \left(\frac{\partial \psi}{\partial v}\right)^2$$

RESULT:

1 S.T

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}$$

Proof:

$$z = x+iy$$

$$\bar{z} = x-iy$$

$$z+\bar{z} = 2x$$

$$\therefore x = \frac{z+\bar{z}}{2}$$

$$y = \frac{z-\bar{z}}{2i}$$

$$y = -\frac{i}{2}(z-\bar{z})$$

$$\frac{\partial x}{\partial z} = 1/2 \qquad \frac{\partial y}{\partial z} = -i/2$$

$$\frac{\partial x}{\partial \bar{z}} = 1/2 \qquad \frac{\partial y}{\partial \bar{z}} = i/2$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial z}$$

$$= \frac{\partial}{\partial x}\,\frac{1}{2} + \frac{\partial}{\partial y}\,(-i/2)$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}$$

$$= \frac{\partial}{\partial x}\,1/2 + \frac{\partial}{\partial y}\,(i/2) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial^2}{\partial z\,\partial \bar{z}} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

$$\therefore \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}$$

If $f(z) = u+iv$ is an analytic fun of z and u is any function of x and y with differential co-efficient of first two order then P.T

(i) $\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2=\left[\left(\frac{\partial\varphi}{\partial u}\right)^2+\left(\frac{\partial\varphi}{\partial v}\right)^2\right]|f'(z)|^2$

(ii) $\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=\left(\frac{\partial^2\varphi}{\partial u^2}+\frac{\partial^2\varphi}{\partial v^2}\right)|f'(z)|^2$

Solution:

(i) $$\frac{\partial\varphi}{\partial x}=\frac{\partial\varphi}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial\varphi}{\partial v}\cdot\frac{\partial v}{\partial x}\longrightarrow (1)$$

$$\frac{\partial\varphi}{\partial y}=\frac{\partial\varphi}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial\varphi}{\partial v}\cdot\frac{\partial v}{\partial y}\longrightarrow (2)$$

By C.R equation

$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x},\quad \frac{\partial v}{\partial y}=\frac{\partial u}{\partial x}$$

$$\therefore \frac{\partial\varphi}{\partial y}=\frac{\partial\varphi}{\partial u}\left(-\frac{\partial v}{\partial x}\right)+\frac{\partial\varphi}{\partial v}\left(\frac{\partial u}{\partial x}\right)$$

$$=-\frac{\partial\varphi}{\partial u}\cdot\frac{\partial v}{\partial x}+\frac{\partial\varphi}{\partial v}\frac{\partial u}{\partial x}\longrightarrow (2)$$

$(1)^2+(2)^2 \Rightarrow$

$$\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2=\left(\frac{\partial\varphi}{\partial u}\right)^2\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial v}\right)^2\left(\frac{\partial v}{\partial x}\right)^2+\cancel{2\frac{\partial\varphi}{\partial u}\cdot\frac{\partial u}{\partial x}\cdot\frac{\partial\varphi}{\partial v}\cdot\frac{\partial v}{\partial x}}$$

$$+\left(\frac{\partial\varphi}{\partial u}\right)^2\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial v}\right)^2\left(\frac{\partial u}{\partial x}\right)^2-\cancel{2\frac{\partial\varphi}{\partial u}\cdot\frac{\partial v}{\partial x}\cdot\frac{\partial\varphi}{\partial v}\cdot\frac{\partial u}{\partial x}}$$

$$=\left(\frac{\partial\varphi}{\partial u}\right)^2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right]+\left(\frac{\partial\varphi}{\partial v}\right)^2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right]$$

$$\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2=\left[\left(\frac{\partial\varphi}{\partial u}\right)^2+\left(\frac{\partial\varphi}{\partial v}\right)^2\right]\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right]\longrightarrow (3)$$

But $f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$

$$|f'(z)|=\sqrt{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2}$$

$$|f'(z)|^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2$$

$\therefore$ From (3)

$$\therefore \left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2=\left(\frac{\partial\varphi}{\partial u}\right)^2+\left(\frac{\partial\varphi}{\partial v}\right)^2|f'(z)|^2$$

(ii) Equation (1) can be written as

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\cdot\frac{\partial v}{\partial x}\longrightarrow (4)$$

$\therefore$ Equ ② can be written as

$$\frac{\partial}{\partial y} = -\frac{\partial v}{\partial x}\cdot\frac{\partial}{\partial u} + \frac{\partial u}{\partial x}\cdot\frac{\partial}{\partial v} \longrightarrow ⑤$$

④ + i⑤ $\Rightarrow$

$$\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\cdot\frac{\partial}{\partial v} + i\left[-\frac{\partial v}{\partial x}\cdot\frac{\partial}{\partial u} + \frac{\partial u}{\partial x}\cdot\frac{\partial}{\partial v}\right]$$

$$= \left(\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}\right)\frac{\partial}{\partial u} + \left(\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\frac{\partial}{\partial v}$$

$$= \left(\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}\right)\frac{\partial}{\partial u} + i\left(\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}\right)\frac{\partial}{\partial v} \quad [i^2=-1$$

$$= \left(\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}\right)\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right) \longrightarrow ⑥$$

④ - i5 $\longrightarrow$

$$\frac{\partial}{\partial x} - i\frac{\partial}{\partial y} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v} - i\left(-\frac{\partial v}{\partial x}\frac{\partial}{\partial u} + \frac{\partial u}{\partial x}\cdot\frac{\partial}{\partial v}\right)$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\frac{\partial}{\partial u} + \left(\frac{\partial v}{\partial x} - i\frac{\partial u}{\partial x}\right)\frac{\partial}{\partial v}$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\frac{\partial}{\partial u} - i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\frac{\partial}{\partial v}$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right) \longrightarrow ⑦$$

⑥ × ⑦ $\Rightarrow$

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\left[\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right]$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)|f'(z)|^2$$

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = \left(\frac{\partial^2\varphi}{\partial u^2} + \frac{\partial^2\varphi}{\partial v^2}\right)|f'(z)|^2$$

③ If $f(z)$ is analytic fun of z P.T
"IUP"

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|\text{Real}(f(z))|^2 = 2|f'(z)|^2$$

Solution

given that $f(z) = u + iv$

real $f(z) = u$

Consider

$$\frac{\partial}{\partial x}(u^2) = 2u\frac{\partial u}{\partial x}$$

$$\frac{\partial^2(u^2)}{\partial x^2} = 2\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial x}\right) + 2u\frac{\partial^2 u}{\partial x^2}$$

$$= 2\left(\frac{\partial u}{\partial x}\right)^2 + 2u\left(\frac{\partial^2 u}{\partial x^2}\right) \longrightarrow (1)$$

$$|||^{ly} \quad \frac{\partial^2(u^2)}{\partial y^2} = 2\left(\frac{\partial u}{\partial y}\right)^2 + 2u\left(\frac{\partial^2 u}{\partial y^2}\right) \longrightarrow (2)$$

$(1) + (2) \Longrightarrow$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u^2 = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + 2u\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]$$

Since u is harmonic $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0$

$\therefore$ The above equation can be written as

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u^2 = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

By C.R condition $\cancel{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$, $\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$

$$\therefore \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u^2 = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]$$

$\therefore$ But $|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u^2 = 2\,|f'(z)|^2$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|\text{Real } f(z)|^2 = 2\,|f'(z)|^2$$

(4) If $f(z) = u+iv$ is a regular (Analytic) function of z

P.T $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^P = P^2\,|f(z)|^{P-2}\,|f'(z)|^2$

W.K.T $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}$ (Result 1)

Here

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\,|f(z)|^P = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\,|f(z)|^P \qquad |z|^2 = z\bar{z}$$

$$= 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\left[f(z)\cdot f(\bar{z})\right]^{P/2}$$

$$= 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\left[\{f(z)\}^{P/2}\cdot\{f(\bar{z})\}^{P/2}\right]$$

$$= 4\frac{\partial}{\partial z}\left[\{f(z)\}^{p/2}\ \frac{p}{2}\ \{\overline{f(z)}\}^{p/2-1}\ \overline{f'(z)}\right]$$

$$= 4\left[\frac{p}{2}\{f(z)\}^{p/2-1} f'(z)\cdot\frac{p}{2}\{\overline{f(z)}\}^{p/2-1}\overline{f'(z)}\right]$$

$$= p^2\{f(z)\ \overline{f(z)}\}^{p/2-1}\{f'(z)\ \overline{f'(z)}\}$$

$$= p^2\{|f(z)|^2\}^{p/2-1}\ |f'(z)|^2$$

$$= p^2\{|f(z)|^2\}^{\frac{p-2}{2}}\ |f'(z)|^2$$

$$= p^2\ |f(z)|^{p-2}\ |f'(z)|^2$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2\ |f(z)|^{p-2}|f'(z)|^2 \ //$$

(5) S.T an analytic fun: with Constant modulus is Constant

solution

Given Let $f(z) = u+iv$

$$|f(z)|^2 = u^2+v^2$$

given that $|f(z)|$ is constant

$$|f(z)| = c \ (\text{say})$$

$$|f(z)|^2 = c^2$$

$$u^2+v^2 = c^2$$

Partially differentiate w.r.t x

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \longrightarrow (2)$$

Part diff w.r.t y

$$2u\frac{\partial u}{\partial y} + 2v\cdot\frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0 \longrightarrow (2)$$

By C.R condition

$$-u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} = 0 \longrightarrow (3)$$

$(2)^2 + (3)^2$

$$u^2\left(\frac{\partial u}{\partial x}\right)^2 + v^2\left(\frac{\partial v}{\partial x}\right)^2 + 2uv\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} +$$

$$u^2\left(\frac{\partial v}{\partial x}\right)^2 + v^2\left(\frac{\partial u}{\partial x}\right)^2 - 2uv\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} = 0$$

$$u^2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] + v^2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] = 0$$

$$(u^2+v^2)\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] = 0$$

$$u^2+v^2 = c^2 \neq 0$$

$$\therefore \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = 0$$

$$|f'(z)|^2 = 0$$

(i.e) $f'(z) = 0$

$\Rightarrow f(z)$ is constant.

(6) If $f(z)$ is analytic function of z S.T

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$

Solution: Given that $f(z) = u+iv$

$\therefore$ Real $f(z) = u$

Consider $\frac{\partial}{\partial x}(u^2) = 2u\frac{\partial u}{\partial x}$

$$\frac{\partial^2}{\partial x^2}(u^2) = 2\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial x} + 2u\frac{\partial^2 u}{\partial x^2}$$

$$= 2\left(\frac{\partial u}{\partial x}\right)^2 + 2u\frac{\partial^2 u}{\partial x^2} \quad \to (1)$$

Ill^{ly} $$\frac{\partial^2}{\partial y^2}(u^2) = 2\left(\frac{\partial u}{\partial y}\right)^2 + 2u\frac{\partial^2 u}{\partial y^2} \quad \to (2)$$

$(1) + (2) \Rightarrow$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u^2 = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + 2u\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

Since u is harmonic

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u^2 = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

By C.R Condition

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(u^2) = 2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right]$$

$$= 2\,|f'(z)|^2 \longrightarrow (3)$$

|||ly

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(v^2) = 2\,|f'(z)|^2 \longrightarrow (4)$$

(3) + (4) =>

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(u^2+v^2) = 4\,|f'(z)|^2$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(|f(z)|^2) = 4\,|f'(z)|^2$$

(7) $w = f(z)$ is a regular function of z P.T

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 0$$

Solution: W.K.T $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\partial\bar{z}}$

$$\therefore \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 4\frac{\partial^2}{\partial z\partial\bar{z}}\log|f'(z)|$$

$$= 4\frac{\partial^2}{\partial z\partial\bar{z}}\left[\frac{1}{2}\log|f'(z)|^2\right] \qquad \log x^n = n\log x$$

$$= 2\frac{\partial^2}{\partial z\partial\bar{z}}\left[\log\{f'(z)\cdot\overline{f'(z)}\}\right] \qquad |z|^2 = z\bar{z}$$

$\overline{f'(z)} = f'(\bar{z})$

$$= 2\frac{\partial^2}{\partial z\partial\bar{z}}\left[\log f'(z)+\log\overline{f'(z)}\right]$$

$$= 2\frac{\partial}{\partial z}\left[0+\frac{1}{\overline{f'(z)}}\overline{f''(z)}\right]$$

$$= 0$$

$$\bar{\partial}\omega = 0$$
$$iz = 0$$
$$i \neq 0,\ z = 0.$$

(i) **Type II**

S.T the function $f(z) = \sin x \cosh y + i \cos x \sinh y$ is Continuous as well as analytic every where

Solution:

Let
$$f(z) = \sin x \cosh y + i \cos x \sinh y$$
$$u + iv = \sin x \cosh y + i \cos x \sinh y$$
$$u = \sin x \cosh y \ ; \qquad v = \cos x \sinh y$$

Here u and v both are rational functions of x and y. whose derivatives are non-zero for all values of x and y

∴ u and v are both Continuous every where

Hence $f(z)$ is also continuous every where

$$\frac{\partial u}{\partial x} = \cos x \cosh y \ ; \qquad \frac{\partial v}{\partial x} = -\sin x \sinh y$$

$$\frac{\partial v}{\partial y} = \sin x\ (+\sinh y); \qquad \frac{\partial v}{\partial x} = \cos x \cosh y$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \implies \cos x \cosh y = \cos x \cosh y$$

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \implies \sin x \sinh y = \sin x \sinh y$$

The given fun is analytic

(ii) C.R Conditions are Satisfies.

Thus the four partial derivatives being Conditions every where and C.R Conditions being satisfies then the given fun is analytic every where

S.T the function $f(z) = xy + iy$ is Continuous every where but it is [illegible] analytic

Solution

$$f(z) = xy + iy$$
$$u + iv = xy + iy$$

Equating real and imaginary part

$$u = xy \qquad v = y$$

$$\frac{\partial u}{\partial x} = y \qquad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = x \qquad \frac{\partial v}{\partial y} = 1$$

Here u and v both are polynomial fun of x and y whose derivative are non-zero for all values of x & y

$\therefore$ u & v are both continuous everywhere

Hence $f(z)$ is continuous everywhere.

C.R equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} \Rightarrow y \neq 1$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{x} \Rightarrow x \neq 0$$

Hence C.R condition are not satisfied

The given fun is not analytic at any Pt.

3. P.T the fun $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$, $(z \neq 0)$

$$f(0) = 0 \qquad , z = 0$$

solution:

is continuous and that C.R condition satisfied at the origin, yet $f'(z)$ does not exist there

solution:

$$u + iv = f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$$

$$= \frac{x^3 + x^3 i - y^3 + y^3 i}{x^2+y^2}$$

$$u + iv = \frac{x^3 - y^3 + i(x^3+y^3)}{x^2+y^2}$$

Equating real and imaginary part

$$u = \frac{x^3-y^3}{x^2+y^2}, \quad v = \frac{x^3+y^3}{x^2+y^2}$$

Here u & v both are rational funs and finite for all values of $z \neq 0$

$\therefore$ u and v are both Continuous every where at all these pts for which $z \neq 0$

Here $f(z)$ is continuous everywhere when $z \neq 0$

At the origin it is given that $f(0) = 0$ (or) at the origin $u=0$, $v=0$

Hence u and v are both Continuous at the origin Consequently $f(z)$ is continuous at the origin. Hence $f(z)$ is Continuous every where

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2)3x^2 - (x^3-y^3)2x}{(x^2+y^2)^2}$$

$$= \frac{3x^4+3x^2y^2-2x^4+2xy^3}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial x} = \frac{x^4+2xy^3+3x^2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(x^2+y^2)3y^2-(x^3+y^3)2y}{(x^2+y^2)^2}$$

$$= \frac{3x^2y^2+3y^4-2x^3y-2y^4}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{y^4-2x^3y+3x^2y^2}{(x^2+y^2)^2}$$

Partial dyn.: $\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0} \frac{x}{x} = 1$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0} \frac{-y}{y} = -1$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0} \frac{x}{x} = +1$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0} \frac{y}{y} = 1$$

By C.R condition

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 1 = 1$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow -1 = -1$$

Hence C.R conditions are satisfy at $z=0$

$$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z}$$

$$= \lim_{z\to 0}\left[\frac{(x^3-y^3)+i(x^3+y^3)}{x^2+y^2} \cdot \frac{1}{x+iy}\right]$$

let $z\to 0$ along $y=x$ then

$$f'(0) = \lim_{x\to 0}\left\{\frac{(x^3-x^3)+i(x^3+x^3)}{x^2+y^2} \frac{1}{x+ix}\right)$$

$$= \lim_{x\to 0}\left[i\frac{\cancel{2}x^3}{\cancel{2}x^2} \frac{1}{x(1+i)}\right]$$

$$= \lim_{x\to 0}\left[\frac{i}{1+i} \times \frac{1-i}{1-i}\right]$$

$$f'(0) = \frac{i+1}{2} \quad \text{—①}$$

again let $z\to 0$ along $y=0$ then

$$f'(0) = \lim_{x\to 0} \frac{x^3+ix^3}{x^2} \cdot \frac{1}{x}$$

$$= \lim_{x\to 0} \frac{\cancel{x^3}(1+i)}{\cancel{x^2}} \frac{1}{\cancel{x}}$$

$$f'(0) = 1+i \quad \rightarrow ②$$

From ① & ②

∴ so we see that $f'(0)$ is not unique.

(ie) The value of $f'(0)$ is not the same as $z\to 0$ along different curve. ($y=x$, $y=0$)

. Hence $f'(z)$ does not exists at the origin

4.

1.Q S.7 $f(z) = \frac{xy^2(x+iy)}{x^2+y^4}$ when $z \neq 0$

$= 0$ when $z = 0$ is not

analytic at $z = 0$

Solution

$$f(z) = \frac{xy^2(x+iy)}{x^2+y^4}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

At the origin

$$f'(0) = \lim_{\Delta z \to 0} \frac{f(z) - f(0)}{\Delta z}$$

$$\boxed{f'(0) = \lim_{\Delta z \to 0} \frac{f(z)}{z}}$$

$$= \lim_{z \to 0} \frac{xy^2(x+iy)}{x^2+y^4} \frac{1}{x+iy}$$

$$f'(0) = \lim_{z \to 0} \frac{xy^2}{x^2+y^4}$$

If $z \to 0$ along the line $y = mx$

$$f(0) = \lim_{x \to 0} \frac{x(mx)^2}{x^2+(mx)^4}$$

$$= \lim_{x \to 0} \frac{m^2x^3}{x^2(1+m^4x^2)} = \lim_{x \to 0} \frac{xm^2}{1+m^4x^2}$$

$$f'(0) = 0 \quad \to (1)$$

Again if $z \to 0$ along the curve $x = y^2$

$$f'(0) = \lim_{y \to 0} \left(\frac{y^4}{y^4+y^4}\right) = \lim_{y \to 0} \frac{y^4}{2y^4}$$

$$= \frac{1}{2} \quad \to (2)$$

$\therefore$ The value of $f'(0)$ is not unique

$\therefore$ $f'(z)$ is not analytic at $z = 0$

5) 2.a Examine the nature of the function

$$f(z) = \frac{x^2y^5(x+iy)}{x^4+y^{10}}, \text{ when } z \neq 0$$

0 when $z=0$ in the region including the origin

Solution:

$$f(z) = \frac{x^2y^5(x+iy)}{x^4+y^{10}}$$

Here we have to find whether C.R conditions are satisfied

$$u+iv = \frac{x^2y^5(x+iy)}{x^4+y^{10}}$$

Equating real and imaginary part

$$u = \frac{x^2y^5(x)}{x^4+y^{10}} \qquad v = \frac{x^2y^5(y)}{x^4+y^{10}}$$

$$= \frac{x^3y^5}{x^4+y^{10}} \qquad v = \frac{x^3y^5}{x^4+y^{10}}$$

At the origin

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{(0)} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{0} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0} \frac{(0-0)}{0} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0} \frac{(0-0)}{0} = 0$$

By C.R Conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 0 = 0$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow 0 = 0.$$

$\therefore$ C.R conditions are satisfy at the origin

$$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z}$$

$$= \lim_{z\to 0} \frac{x^2y^5(x+iy)}{x^4+y^{10}} \frac{1}{x+iy}$$

4. S.T $f(z) = \frac{xy^2(x+iy)}{x^2+y^4}$, when $z \neq 0$

$= 0$, when $z = 0$

is not analytic at $z = 0$

~~Solution $f(z) = \frac{xy^2(x+iy)}{x^2+y^4}$~~

~~$u+iv = \frac{xy^2(x+iy)}{x^2+y^4}$~~

If $z \to 0$ along the radius vector $y = mx$

$$f'(0) = \lim_{x\to 0} \frac{x^3(mx)^5}{x^4(mx)^{10}} = \lim_{x\to 0} \frac{x^4 x^3 m^5}{x^4(1+m^{10}x^6)}$$

$$= \lim_{x\to 0} \frac{x^3 m^5}{1+m^{10}x^6}$$

$$= 0 \longrightarrow (1)$$

Again if $z \to 0$ along the curve $y^5 = x^2$ we have

$$f'(0) = \lim_{x\to 0} \frac{x^2 x^2}{x^4+x^4}$$

$$= \lim_{x\to 0} \frac{x^4}{2x^4}$$

$$f'(0) = 1/2 \longrightarrow (2)$$

we conclude that
From (1) & (2), $f'(0)$ is not unique.
Hence $f(z)$ is not analytic at the origin eventhough C.R equations are satisfied at the origin

6(b) 24/11
(5) S.T $f(z) = \sqrt{|xy|}$ satisfies C.R equations at the origin that is not analytic at that pt

sol
$f(z) = u(x,y) + iv(x,y)$

$f(z) = \sqrt{|xy|}$

$u(x,y) = \sqrt{|xy|}$

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0} \frac{(0-0)}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0)-u(0,0)}{x} = \lim \frac{0-0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y} = \lim \frac{0-0}{y} = 0$$

By C.R equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0$

Hence C.R equation are statisfy at origin

Further $f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{z\to 0} \frac{\sqrt{|xy|}}{x+iy}$

Let $z\to 0$ along the line $y = mx$. we have

$$f'(0) = \lim_{x\to 0} \frac{\sqrt{|x\cdot mx|}}{x+imx}$$

$$f'(0) = \lim_{x\to 0} \frac{\cancel{x}\sqrt{|m|}}{\cancel{x}(1+im)} = \frac{\sqrt{|m|}}{(1+im)}$$

Here the derivative depends on m. $\therefore$ $f'(0)$ is not unique

Again ~~at~~ $\not\exists$ ~~At~~ also

$f(z)$ is not analytic at the origin

although C.R equation are satisfy at origin

(9) S.T the function (6)

If $u = \frac{1}{2}\log(x^2+y^2)$ find its conjugate harmonic

Sol

Let v be the conjugate harmonic function

$$dv = \frac{\partial v}{\partial x}\cdot dx + \frac{\partial v}{\partial y} dy$$

By C.R equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

$$\frac{\partial u}{\partial x} = \frac{1}{\cancel{2}} \frac{1}{(x^2+y^2)^2} \cancel{2}x \qquad \frac{\partial u}{\partial y} = \frac{y}{(x^2+y^2)}$$

$$= \frac{x}{(x^2+y^2)^2}$$

$$dv = -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

$\int^{ing}$ both sides

$$dv = \frac{x\,dy - y\,dx}{(x^2+y^2)}$$

$$v = \tan^{-1}(y/x) + c$$

(8) Apr 98 S.T the fun $u = \cos x \cosh y$ is harmonic 5(1)

solution

$$u = \cos x \cosh y$$

$$\frac{\partial u}{\partial x} = -\sin x \cosh y \qquad \frac{\partial u}{\partial y} = \cos x \sinh y$$

$$\frac{\partial^2 u}{\partial x^2} = -\cos x \cosh y \qquad \frac{\partial^2 y}{\partial y^2} = \cos x \cosh y$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\therefore u$ is harmonic

v - may not be analytic

$u = y^3 - 3x^2y$ is harmonic and determine its conjugate harmonic and find the corresponding fn $f(z)$ interms of z

Solution

$$u = y^3 - 3x^2y$$

$$\frac{\partial u}{\partial x} = -6xy \qquad \frac{\partial u}{\partial y} = 3y^2 - 3x^2$$

$$\frac{\partial^2 u}{\partial x^2} = -6y \qquad \frac{\partial^2 u}{\partial y^2} = 6y$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6y + 6y = 0$$

u satisfy the Laplace equation.

$\therefore$ u is a harmonic

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

By C.R equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$$= \cancel{6xy\,dx}\,(3x^2 - 3y^2)dx - 6xy\,dy = 3x^2dx - d(3xy^2)$$

$\int$ both sides

$$V = \cancel{\frac{3x^3}{3} - 3y^2x - 3\,6x\frac{y^2}{2}} \qquad = \frac{3x^3}{3} - 3xy^2 = x^3 - 3xy^2$$

$$V = x^3 - 3y^2x - \cancel{3xy^2} + c$$

$= x^3 - 3xy^2 + c$, which is the Conjugate harmonic of u

$$f(z) = u + iV$$

$$= y^3 - 3x^2y + i(x^3 - 3xy^2) + c$$

$$= i[x^3 - 3xy^2 + i3x^2y - iy^3] + ci - \cancel{}$$

$$= i[x + iy]^3 + c_1$$

$f(z) = iz^3 + c_1$

By Milne thomson method

$$f(z) = \int [\phi_1(z,0) - i\,\phi_2(z,0)]\, dz + c_1$$

where $\phi_1(z,0) = \frac{\partial u}{\partial x}$, $\phi_2(z,0) = \frac{\partial u}{\partial y}$

$= 3y^2 - 3x^2$

$= -6xy$ $\quad = -3z^2$

$= 0$

$$f(z) = \int 0 - i(-3z^2)\, dz + c$$

$$= i\int 3z^2 dz + c$$

$$= i\,\frac{3z^3}{3} + c$$

$$f(z) = iz^3 + c_1$$

11) If $u = x^3 - 3xy^2$ S.T there exist a function $V(x,y)$ such that $\omega = u + iv$ is an analytic in a finite region

Solution: $u = x^3 - 3xy^2$

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$ $\quad \frac{\partial u}{\partial y} = -6xy$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

By C.R equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

$$= +6xy\, dx + 3x^2 - 3y^2 dy$$

$$= -3y^2 dy + d(3x^2 y)$$

$d(3x^2y)$

$\int$ing both sides $\quad 6xy dx + 3x^2 dy$

$$v = -\frac{3y^3}{3} + 3x^2y + c$$

$$= \ldots^2 y - y^3 + c$$

$$f(z) = u + iv$$

$$= x^3 - 3xy^2 + i3x^2y - iy^3 + c$$

$$= (x+iy)^3 + c$$

$$f(z) = z^3 + c$$

By Milne Thomson method:

$$f(z) = \int [\phi_1(z,0) - i\,\phi_2(z,0)]\,dz + c.$$

$\phi_1(x,y) = \frac{\partial u}{\partial x} = 3x^2 - 3y^2$, $\quad \phi_2(x,y) = \frac{\partial u}{\partial y} = -6xy$

$$= \int 3z^2\,dz + c.$$

$$= \frac{3z^3}{3} + c$$

$$f(z) = z^3 + c$$

Type-3 using Milne Thomson method

$$f(z) = \int [\phi_1(z,0) - i\,\phi_2(z,0)]\,dz + c_1$$

where $\phi_1(x,y) = \frac{\partial u}{\partial x}$ $\quad \phi_2(x,y) = \frac{\partial u}{\partial y}$

$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$f'(z) = u_x - iu_y$

$= v_y + iv_x$

$$f(z) = \int [\psi_1(z,0) + i\,\psi_2(z,0)]\,dz + c_2$$

$$\frac{\partial v}{\partial y} = \psi_1(x,y) \qquad \frac{\partial v}{\partial x} = \psi_2(x,y)$$

Q) find an analytic fun whose imaginary part is $3x^2y - y^3$

Solution: given that $v(x,y) = 3x^2y - y^3$

$$\frac{\partial v}{\partial x} = 6xy = \psi_2(x,y) \qquad \frac{\partial v}{\partial y} = 3x^2 - 3y^2 = \psi_1(x,y)$$

$\therefore$ By Milne Thomson method

$$f(z) = \int [\psi_1(z,0) + i\,\psi_2(z,0)]\,dz + c$$

$\therefore \psi_1(z,0) = 3z^2$

$\psi_2(z,0) = 0$

$$f(z) = \int 3z^2 + i0\,dz + c$$

$$f(z) = z^3 + c_1$$

(2) If $u(x,y) = e^x(x\cos y - y\sin y)$ find $f(z)$ so that $f(z)$ is analytic.

Solution

$$u(x,y) = e^x(x\cos y - y\sin y)$$

$$\frac{\partial u}{\partial x} = \cos y(xe^x + e^x) - e^x y\sin y$$
$$= e^x[x\cos y + \cos y - y\sin y]$$
$$= \phi_1(x,y)$$

$$\frac{\partial u}{\partial y} = e^x(x\sin y - \sin y - y\cos y)$$
$$= e^x[-x\sin y - \sin y - y\cos y]$$
$$= \phi_2(x,y)$$

$$\phi_1(z,0) = (\cancel{ze^z + e^z})$$
$$= e^z(z+1)$$

$$\phi_2(z,0) = e^z(0) = 0$$

$$f(z) = \int[\phi_1(z,0) - i\phi_2(z,0)]\,dz + c$$
$$= \int(z+1)e^z\,dz + c$$
$$= \left[\int ze^z + \int e^z\right]dz + c$$
$$= \int z\,d(e^z) + \int e^z dz + c$$
$$= ze^z - \int e^z dz + \int e^z dz + c_1$$
$$f(z) = ze^z + c_1$$

$u = z \quad dv = e^z dz$

$\int u\,dv = uv - \int v\,du$

$= ze^z - \int e^z dz$

If a function is analytic s.t. it is independent of $\bar{z}$.

To prove that Let the function be $w = f(z)$

Solution: $f(z)$ is independent of $\bar{z}$, we must have $\frac{dw}{d\bar{z}} = 0$

$$w = f(z)$$

$$z = x + iy, \quad \bar{z} = x - iy$$

$$\frac{dw}{d\bar{z}} = 0$$

$$x = \frac{1}{2}(z + \bar{z}), \quad y = \frac{1}{2i}(z - \bar{z}) = \frac{1}{2}(-i)(z - \bar{z})$$

$$= \frac{1}{2} \quad \frac{\partial y}{\partial \bar{z}} = +\frac{i}{2}$$

But

$$\frac{d\omega}{d\bar{z}} = \frac{du}{d\bar{z}} + i\frac{dv}{d\bar{z}}$$

$$\frac{du}{d\bar{z}} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}$$

$$= \frac{\partial u}{\partial x}(1/2) + \frac{\partial u}{\partial y}(i/2) \longrightarrow (2)$$

$$\frac{dv}{d\bar{z}} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}$$

$$= \frac{\partial v}{\partial x}(1/2) + \frac{\partial v}{\partial y}(i/2) \longrightarrow (3)$$

using (2) & (3) in (1)

$$\frac{d\omega}{d\bar{z}} = \frac{1}{2}\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right] + \frac{i}{2}\left[\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right]$$

$$= \frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] + \frac{i}{2}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right]$$

By C.R condition.

$$= \frac{1}{2}(0) + \frac{i}{2}(0)$$

$$= 0$$

$$\frac{d\omega}{d\bar{z}} = 0$$

Hence ω is independent of $\bar{z}$.

4) S.T the fun $f(z) = |z|^2$ is differentiable only at the origin. [P.T the fun $f(z) = |z|^2$ is not analytic nowhere even though it has derivative at the origin]

Solution.

given that $f(z) = |z|^2 \rightarrow (1)$

(ie) $f(z) = x^2+y^2 \longrightarrow (2)$

$u+iv = x^2+y^2$

$u = x^2+y^2 \qquad v = 0$

$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = 0$

$\frac{\partial u}{\partial y} = 2y \qquad \frac{\partial v}{\partial y} = 0$

If $f(z)$ is differentiable

C.R condition

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

(ie) $\therefore 2x = 0$

$\rightarrow x = 0$

||ly $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

ie $2y = 0$

$\Rightarrow y = 0$

$\therefore$ C.R conditions are satisfyed only when $x = 0,\ y = 0$.

Hence the given fun is diffentiable only at the origin

S.T the fun $\bar{z}$ is no where differntiable

solution

given that $f(z) = \bar{z}$

$u + iv = x - iy$

Equating real and imaginary

$u = x \qquad V = -y$

$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial y} = -1$

$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial x} = 0$

$$\frac{\partial u}{\partial x} = +\frac{\partial v}{\partial y}$$

$$1 \neq -1$$

Hence C.R equations are not saitisfyed anywhere

Hence $f(z) = \bar{z}$ is no where differentiable.

S.T the function $u(x,y) = e^x \cos y$ is harmonic determine its harmonic conjugate such that $f(z)$ is analytic

$U(x,y) = e^x \cos y$

$u_x = e^x \cos y \qquad u_y = e^x(-\sin y)$

$u_{xx} = e^x \cos y \qquad u_{yy} = -e^x \cos y$

$$u_{xx} + u_{yy} = 0$$

satisfy the Laplace equation and $u(x,y)$ is a harmonic

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$\int$ both sides $= e^x \sin y\, dx + e^x \cos y\, dy$ $= d(e^x \sin y)$

$dv = e^x \cancel{\ }\, dx + e^x (\cancel{\ })dy$ $dv = d e^x \sin y$

$v = e^x \sin y + c$

$v = e^x [\cancel{\ } + \sin y]$

$$f(z) = u + iv$$

$$= e^x \cos y + i e^x [\cancel{\cos y} + \sin y] + c$$

$$= e^x [\cos y + i \sin y] + \cancel{i e^x \cos y}$$

$$= e^x \cdot e^{iy} + \cancel{i e^x \cos y} + c$$

$$= e^{x+iy} + c$$

$$f(z) = e^z + c$$

(7) Let $f(z) = u+iv$ be analytic fun in a domain D that P.T $f(z)$ is constant in the D If any of the following conditions holds.

(i) Real $(f(z)) = u$, a constant

(ii) Imag $(f(z)) = v$ a constant

(iii) $\operatorname{Arg}[f(z)]$ = a constant

(iv) S.T an analytic fnd with const modulus is constant (Problem 8)

~~Solution:~~

case (i)

Given Real $[f(z)] = u$, a constant

$$u = \text{a constant}$$

$$\frac{\partial u}{\partial x} = 0 = \frac{\partial u}{\partial y} \rightarrow (1)$$

But $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

By C.R equation

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = 0$$

$$= 0 - i(0) = 0$$

$$f'(z) = c$$

$$f(z) = c$$

$\text{Imag}[f(z)] = v$

$v =$ a constant

$\frac{\partial v}{\partial x} = 0 = \frac{\partial v}{\partial y}$

But $f'(z) = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial x}$

(By C.R condition)

$$f'(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$
$$= 0 + i0$$
$$f'(z) = 0$$
$$\Rightarrow f(z) = \text{a constant}$$

(iii) $\text{Arg}[f(z)] = \text{Constant}$ $\qquad \theta = \tan^{-1}(y/x)$

(a) $\text{Arg}[f(z)] = \tan^{-1}(v/u)$

$\tan^{-1}(v/u) = c$, a Constant

$\frac{v}{u} = \tan c$

$\frac{u}{v} = \frac{1}{\tan c} = \cot c$

$u = v\cot c$

If $\cot c = k$ then

$u = vk$

$(u - vk) = 0$, unless v is identically zero

now Consider

$(1+ik) f(z) = (1+ik)(u+iv)$

$= u + iku + iv - kv$

$= (u - kv) + i(ku+v)$ — (1)

Here $u-kv$ is a real part of $(1+ik)(u+iv)$

From (1) $\therefore$ It follows that $(1+ik) f(z)$ is constant,

Since $(1+ik)$ is constant.

$\therefore$ we conclude

5) S.T the fun $u = \sin x \cosh y + 2 \cos x \sinh y + x^2 - y^2 + 4xy$ S.T satisfies laplace equation and determine the fun $f(z)$

Solution:

$$f(z) = u + iv$$

$$u = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$$

$$\frac{\partial u}{\partial x} = \cos x \cosh y - 2\sin x \sinh y + 2x + 4y = \phi_1(x,y)$$

$$\frac{\partial u}{\partial y} = \sin x \sinh y + 2\cos x \cosh y - 2y + 4x \quad \phi = (x,y)$$

By Milne's Thomson Method.

$$f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\, dz + c$$

$$= \int [\cos z + 2z - i(2\cos z + 4z)]\, dz + c$$

$$= \int [\cos z + 2z - i\, 2(\cos z + 2z)]\, dz + c$$

$$= \int \cos z + 2z\, (1-2i)\, dz + c$$

$$= (1-2i)\left[\int \cos z\, dz + \int 2z\, dz + c\right]$$

$$= (1-2i)\left(\sin z + \cancel{2}\frac{z^2}{2} + c_1\right)$$

$$= (1-2i)(\sin z + z^2) + c_1$$

$$\frac{\partial^2 u}{\partial x^2} = -\sin x \cosh y - 2\cos x \sinh y + 2$$

$$\frac{\partial^2 u}{\partial y^2} = \sin x \cosh y + 2\cos x \sinh y - 2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\sin x \cosh y - 2\cos x \sinh y + 2 + \sin x \cosh y + 2\cos x \sinh y - 2$$

$$= 0$$

$\therefore$ u satisfy the L.E

$\therefore$ u is harmonic

(6) $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

Sol

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x \quad , \quad \frac{\partial u}{\partial y} = -6xy - 6y$$

$$\phi_1(x,y) \qquad\qquad = \phi_2(x,y)$$

$$\frac{\partial^2 u}{\partial x^2} = 6x + 6 \quad , \quad \frac{\partial^2 u}{\partial y^2} = -6x - 6$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x + 6 - 6x - 6$$

$$= 0$$

$\therefore$ Laplace equation are satisfied

By Milne thomoason method

$$f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\, dz + c$$

$$= \int (3z^2 + 6z)\, dz - i(0)\, dz + c$$

$$= \frac{\cancel{3}z^3}{\cancel{3}} + \overset{3}{\cancel{6}}\frac{z^2}{\cancel{2}} + c_1$$

$$= z^3 + 3z^2 + c_1$$

$$f(z) = z^3 + 3z^2 + c_1$$

Another method using Milne thomson method

If $f(z) = u+iv$ is an analytic function of z & $u-v = e^x(\cos y - \sin y)$ find $f(z)$ interms of z

Solution:

$$u+iv = f(z)$$
$$iu - v = if(z)$$
$$(u-v)+i(u+v) = (1+i)f(z)$$
$$U + iV = F(z)$$

where $U = u-v$, $V = u+v$

$$F(z) = (1+i)f(z)$$

given that $f(z) = u+iv$ is an analytic fun.

$\therefore$ $U+iV = F(z)$, is also an analytic fun

$$\therefore\ u-v = U = e^x(\cos y - \sin y)$$

$$\frac{\partial U}{\partial x} = e^x(\cos y - \sin y) = \phi_1(x,y)$$

$$\frac{\partial U}{\partial y} = e^x(-\sin y - \cos y) = \phi_2(x,y)$$

By Milne thomson method

$$F(z) = \int[\phi_1(z,0) - i\,\phi_2(z,0)]\,dz + c$$
$$= \int[e^z(1-0) - i\,e^z(-1)]\,dz + c$$
$$= \int e^z(1+i)\,dz + c$$
$$= (1+i)\int e^z dz + c$$
$$(1+i)f(z) = (1+i)e^z + c_1$$
$$\boxed{f(z) = e^z + c_2}$$

---

(2) If $(u-v) = (x-y)(x^2+4xy+y^2)$ find $f(z)$ interms

Solution

$$u+iv = f(z)$$
$$iu - v = if(z)$$
$$(u-v)+i(u+v) = (1+i)f(z)$$

$U + iV = F(z)$, $F(z) = (1+i) f(z)$

where $U = u - v$, $V = u + v$

given that $f(z) = u + iv$ is an analytic fun

$\therefore U + iV = F(z)$ is also an analytic function

$u - v = U = (x-y)(x^2+4xy+y^2)$

$= x^3 + 4x^2y + xy^2 - yx^2 - 4xy^2 - y^3$

$\frac{\partial U}{\partial x} = 3x^2 + 8xy + y^2 - 2xy - 4y^2 = \phi_1(x,y)$

$\frac{\partial U}{\partial y} = 4x^2 + 2xy - x^2 - 8xy - 3y^2 = \phi_2(x,y)$

By M.T.M

$F(z) = \int[\phi_1(z,0) - i\,\phi_2(z,0)]\,dz + c$

$= \int(3z^2 - i\,3z^2)\,dz + c$

$= \cancel{3}\frac{z^3}{\cancel{3}} - i\,\frac{\cancel{3}z^3}{\cancel{3}} + c_1$

$(1+i) f(z) = z^3(1-i) + c_1$

$f(z) = \frac{1-i}{1+i} z^3 + c_2$

$= \frac{1-i}{1+i} \times \frac{1-i}{1-i} z^3 + c_2$

$= -\frac{2i}{2} z^3 + c_2$

$\boxed{f(z) = -i z^3 + c_2}$

(3) If $u + v = \frac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$ find $f(z)$ interms of z

Solution

$u + iv = f(z)$

$iu - v = i f(z)$

$(u-v) + i(u+v) = (i+1) f(z)$

$U + iV = F(z)$ $U = u - v$ $V = u + v$

$F(z) = f(z)(1+i)$

$$u+v = V = \frac{2\sin 2x}{2\cosh 2y - 2\cos 2x}$$

$$= \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

$$\frac{\partial v}{\partial y} = \frac{(\cosh 2y - \cos 2x)(0) - \sin 2x \, 2\sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2} = \varphi_1(x,y)$$

$$\frac{\partial v}{\partial x} = \frac{(\cosh 2y - \cos 2x) 2\cos 2x - \sin 2x \, (2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\cosh 2y \cos 2x - 2 \, (1)}{(\cosh 2y - \cos 2x)^2} = \varphi_2(x,y)$$

By M.T.M

$$F(z) = \int [\varphi(z,0) + i\varphi_2(z,0)] \, dz + C$$

$$= \int 2(0) + i \frac{(2\cos 2z - 2)}{(1-\cos 2z)^2} dz + C$$

$$= 2i \int \frac{\cos 2z - 1}{(1-\cos 2z)^2} dz + C$$

$$= -2i \int \frac{1-\cos 2z}{(1-\cos 2z)^2} dz + C$$

$$= -2i \int \frac{dz}{1-\cos 2z} + C$$

$$= -2i \int \frac{dz}{2\sin^2 z} + C$$

$$= -i \int \operatorname{cosec}^2 z \, dz + C$$

$$(1+i) f(z) = +i \cot z + C_1$$

$$f(z) = \frac{i\cot z}{1+i} + C_2$$

$$= \frac{i}{(1+i)} \frac{(1-i)}{(1-i)} \cot z + C_2$$

$$f(z) = \frac{i+1}{2} \cot z + C_2$$

Q (X) 17 Sep (4)

$u-v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}}$ find $f(z)$ subject to the condition $f(\pi/2) = 0$

Solution

$u + iv = f(z)$

$\underline{iu - v = i f(z)}$

$u - v + i(u+v) = (1+i) f(z)$

$U + iV = F(z)$ where $U = u - v$, $V = u + v$

$F(z) = f(z)(1+i)$

$u - v = U = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - e^{-y} - e^{+y}}$

$\sinh y - \cosh y$

$= \dfrac{e^{y} - e^{-y}}{2} - \dfrac{e^{y} + e^{-y}}{2}$

$= \dfrac{e^{y} - e^{-y} - e^{y} - e^{-y}}{2}$

$= -2\dfrac{e^{-y}}{2}$

$= -e^{-y}$

$= \dfrac{\cos x + \sin x + \sinh y - \cosh y}{2\cos x - 2\cosh y}$

$= \dfrac{1}{2}\left[\dfrac{(\cos x - \cosh y) + (\sin x + \sinh y)}{\cos x - \cosh y}\right]$

$U = \dfrac{1}{2}\left[1 + \dfrac{\sin x + \sinh y}{\cos x - \cosh y}\right] \longrightarrow (1)$

$\dfrac{\partial U}{\partial x} = \dfrac{1}{2} \dfrac{(\cos x - \cosh y)(\cos x) - (\sin x + \sinh y)(-\sin x)}{(\cos x - \cosh y)^2}$

$= \dfrac{1}{2}\left[\dfrac{-\cos x \cosh y + \cos^2 x + \sin^2 x + \sin x \sinh y}{(\cos x - \cosh y)^2}\right]$

$= \dfrac{1}{2} \dfrac{[1 - \cos x \cosh y + \sin x \sinh y]}{(\cos x - \cosh y)^2}$

$\dfrac{\partial U}{\partial y} = \dfrac{1}{2} \dfrac{(\cos x - \cosh y)(+\sinh y) - (\sin x + \sinh y)(-\sinh y)}{(\cos x - \cosh y)^2}$

$= \dfrac{1}{2}\left[\dfrac{+\sinh y \cos x - \cosh^2 y + \sin x \sinh y + \sinh^2 y}{(\cos x - \cosh y)^2}\right]$

$= \dfrac{1}{2}\left[\dfrac{\cos x \cosh y + \sin x \sinh y - 1}{(\cos x - \cosh y)^2}\right]$

$-\sinh^2 y + \cosh^2 y = 1$

$\sin^2 x + \cos^2 x = 1$

By Milne's thomoson method

$$F(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\, dz + c$$

$$= \frac{1}{2} \int \left[\frac{1-\cos z}{(\cos z - 1)^2} - i \frac{\cancel{\cos z - 1}}{(\cos z - 1)^{\cancel{2}}} \right] dz + c$$

$$= \frac{1}{2} \int \left[\frac{-(\cos z - 1)}{(\cos z - 1)^2} - i \frac{1}{\cos z - 1} \right] dz + c$$

$$= \frac{1}{2} \int -\frac{1}{(\cos z - 1)} - i \frac{1}{(\cos z - 1)}\, dz + c$$

$$= -\frac{1}{2} \int \frac{1}{\cos z - 1} (1+i)\, dz + c$$

$$= -\frac{(1+i)}{2} \int \frac{1}{\cos z - 1}\, dz + c$$

$$= -\frac{(1+i)}{2} \int \frac{1}{-2\sin^2 z/2}\, dz + c$$

$$\left[\begin{aligned} \cos z &= \cos^2 z/2 - \sin^2 z/2 \\ &= 1 - \sin^2 z/2 - \sin^2 z/2 \\ &= 1 - 2\sin^2 z/2 \\ \cos z - 1 &= -2\sin^2 z/2 \end{aligned} \right.$$

$$= +\frac{(1+i)}{4} \int \operatorname{cosec}^2 z/2\, dz + c$$

$$= -\frac{(1+i)}{\cancel{4}\,2} \cancel{(2)} \cot z/2 + c_1$$

$$F(z) = -\frac{(1+i)}{2} \cdot \cot(z/2) + c_1$$

$$(1+i) f(z) = -\frac{(1+i)}{2} \cot(z/2) + c_1$$

$$f(z) = -\frac{1}{2} \cot z/2 + c_2 \longrightarrow (2)$$

The given condition is. when $z = \pi/2$, $f(\pi/2) = 0$

$$0 = -\frac{1}{2} \cot(\pi/4) + c_2$$

$$= -\frac{1}{2}(1) + c_2$$

$$c_2 = 1/2$$

$$f(z) = -\frac{1}{2} \cot z/2 + \frac{1}{2}$$

$$= \frac{1}{2}(1 - \cot z/2)$$

(5) If $u-v = \dfrac{e^y - \cos x + \sin x}{\cosh y - \cos x}$ find $f(z)$ subject to the condition $f(\pi/2) = \frac{3-i}{2}$

<u>solution:</u>

$$u + iv = f(z)$$

$$\underline{iu - v = if(z)}$$

$$u - v + i(u+v) = (1+i)f(z)$$

$$U + iV = F(z) \quad \text{where } U = u - v$$

$$V = u + v$$

$$F(z) = (1+i)f(z)$$

$$U = u - v = \frac{e^y - \cos x + \sin x}{\cosh y - \cos x}$$

$\cosh y + \sinh y$

$= \frac{e^y + e^{-y}}{2} + \frac{e^y - e^{-y}}{2}$

$= \frac{1}{2}[e^y + e^{-y} + e^y - e^{-y}]$

$= e^y$

$$= \frac{\cosh y + \sinh y - \cos x + \sin x}{\cosh y - \cos x}$$

$$= \frac{\cosh y - \cos x + \sinh y + \sin x}{\cosh y - \cos x}$$

$$= \left(1 + \frac{\sinh y + \sin x}{\cosh y - \cos x}\right) \longrightarrow (1)$$

$$\frac{\partial U}{\partial x} = 0 + \frac{(\cosh y - \cos x)(\cos x) - (\sinh y + \sin x)(\sin x)}{(\cosh y - \cos x)^2}$$

$$= \frac{\cos x \cosh y - \cos^2 x - \sinh y \sin x - \sin^2 x}{(\cosh y - \cos x)^2}$$

$$= \frac{\cos x \cosh y - \sinh y \sin x - 1}{(\cosh y - \cos x)^2} = \phi_1(x,y)$$

$$\frac{\partial U}{\partial y} = \frac{(\cosh y - \cos x)(\cosh y) - (\sinh y + \sin x)(\sinh y)}{(\cosh y - \cos x)^2}$$

$$= \frac{\cosh^2 y - \cos x \cosh y - \sinh^2 y - \sinh y \sin x}{(\cosh y - \cos x)^2}$$

$$= \frac{1 - \cos x \cosh y - \sinh y \sin x}{(\cosh y - \cos x)^2} = \phi_2(x,y)$$

By M.T.M

$$F(z) = \int [\phi_1(z,0) - i\,\phi_2(z,0)]\,dz + c$$

$$= \int \left[\frac{\cos z - 1}{(1-\cos z)^2} - i\,\frac{1 - \cos z}{(1-\cos z)^2}\right] dz + c$$

$$= -\int \left[\frac{(1-\cos z)}{(1-\cos z)^{\cancel{2}}} + i\,\frac{(1-\cos z)}{(1-\cos z)^{\cancel{2}}}\right] dz + c$$

$$= -\int \frac{1}{1-\cos z}(1+i)\, dz + c$$

$$= -(1+i)\int \frac{1}{1-\cos z}\, dz + C$$

$$= -(1+i)\int \frac{dz}{2\sin^2 z/2}\, \cancel{dz} + C$$

$$= -\frac{(1+i)}{2}\int \operatorname{cosec}^2 z/2\, dz + C$$

$$= + \frac{(1+i)}{\cancel{2}} \cot z/2\,(\cancel{2}) + C_1$$

$$F(z) = +(1+i)\cot z/2 + C_1$$

$$(1+i) f(z) = +(1+i)\cot z/2 + C_1$$

$$f(z) = + \cot z/2 + C_2$$

By the given condition $z = \pi/2$, $f(\pi/2) = \frac{3-i}{2}$

$$f(\pi/2) = \cot \pi/4 + C_2$$

$$\frac{3-i}{2} = 1 + C_2$$

$$C_2 = \frac{3-i}{2} - 1$$

$$= \frac{3-i-2}{2}$$

$$C_2 = \frac{1-i}{2}$$

$$\boxed{\therefore f(z) = \cot z/2 + \left(\frac{1-i}{2}\right)}$$

6) H.W. If $\overset{u-v}{f(z)} = \frac{x}{x^2+y^2}$ find $f(z)$ subject to a condition $f(1) = 1$

Ans: $f(z) = \frac{1+i}{2z} + \frac{1-i}{2}$

$$u + iv = f(z)$$
$$iu - v = i f(z)$$
$$u - v + i(u+v) = (1+i) f(z)$$

UNIT - IV

Transformation

1. Discuss the transformation $w = \frac{1}{z}$.

Solution:

The transformation $w = \frac{1}{z}$ is a 1-1 correspondence between the non-zero points of z-plane and w-plane.

$\therefore$ The transformation is conformal at all points except at $z = 0$.

$\because \frac{dw}{dz} = \frac{-1}{z^2}$ and we have $w = \frac{1}{z}$ —(1)

$$u + iv = \frac{1}{x+iy} \times \frac{x-iy}{x-iy}$$

$$= \frac{x-iy}{x^2+y^2}$$

$$u + iv = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

$u = \frac{x}{x^2+y^2}$ —(2) , $v = \frac{-y}{x^2+y^2}$ —(3)

Squaring and adding (2) & (3), we have

$$u^2+v^2 = \frac{x^2+y^2}{(x^2+y^2)^2} \Rightarrow \frac{1}{x^2+y^2}$$

$$\Rightarrow x^2+y^2 = \frac{1}{u^2+v^2} \longrightarrow (4)$$

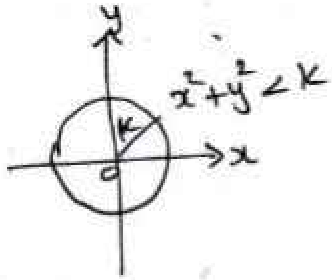
Case(i): Consider the circle with centre at the origin in the xy-plane (z-plane)

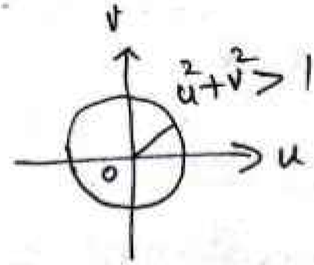
The circle will be of the form $x^2+y^2 = K$ in z-plane

$\Rightarrow u^2+v^2 = \frac{1}{x^2+y^2} = \frac{1}{K}$ in the w-plane

(ie) The unit circle with centre at the origin in z-plane maps into the unit circle in w-plane with centre at origin.

Also circles lying outside the unit circle $x^2+y^2 = K$ in z-plane transform into the circle within the unit circle in the w-plane.





Case (ii)

Any circle in the z-plane is of the form

$x^2+y^2+2gx+2fy+c=0$

Part (i) Let $c=0$ then the circle in the z-plane passes through the origin.

$\therefore$ we have

$$\frac{1}{u^2+v^2}+\frac{2gu}{u^2+v^2}-\frac{2fv}{u^2+v^2}=0 \quad \text{—} (1)$$

multiplying by u^2+v^2 we have

$$1+2gu-2fv=0$$

This represent a st. line in w-plane.

Part (ii) Let $c\neq 0$, $x^2+y^2+2gx+2fy+c=0$

$\times$ u^2+v^2 $\quad \dfrac{1}{u^2+v^2}+\dfrac{2gu}{u^2+v^2}-\dfrac{2fv}{u^2+v^2}+c=0$

$\div$ c $\quad 1+2gu-2fv+c(u^2+v^2)=0$

$$\frac{1}{c}+\frac{2gu}{c}-\frac{2f}{c}v+u^2+v^2=0$$

which also represent a st. line.

Case (iii) Consider a St. line $lx+my+n=0$ in z-plane

$\therefore$ The corresponding transformation in w-plane

is given by $\quad lx+my+n=0$

$$\frac{lu}{u^2+v^2}-\frac{mv}{u^2+v^2}+n=0$$

$$lu-mv+n(u^2+v^2)=0$$

$$lu-mv=-n(u^2+v^2)$$

This is a circle passing through the origin.

Part (i)
when $n=0$ the st. line $lx+my+n=0$ is transformed into $\frac{lu}{u^2+v^2} - \frac{mv}{u^2+v^2} = 0$

$$lu - mv = 0$$

which is a st. line through the origin.

Part (ii). Let $x=k$ then we have

$$x = \frac{u}{u^2+v^2}, \quad k = \frac{u}{u^2+v^2}$$

$$(u^2+v^2)k = u$$

$$u^2k + v^2k - u = 0$$

$$\div k \quad u^2+v^2 - \frac{u}{k} = 0 \Rightarrow u^2+v^2 = \frac{u}{k}$$

This is a circle through the origin and v-axis is tangent.

Part (iii) Let $y=c$

$$y = \frac{-v}{u^2+v^2}$$

$$y = c = \frac{-v}{u^2+v^2}$$

$$(u^2+v^2)c = -v$$

$$u^2c + v^2c + v = 0$$

$$\div c \quad u^2+v^2+v/c = 0$$

This is a circle through the origin and the u-axis is tangent.

Part (iv) Let the line in y-axis $x=0$

$$x = u/u^2+v^2$$

$$\Rightarrow 0 = \frac{u}{u^2+v^2}$$

$\therefore$ The line is v-axis

$|||^{ly}$ Let the line be x-axis

i.e; $y = 0$

$$y = \frac{-v}{u^2+v^2}$$

$$0 = \frac{-v}{u^2+v^2}$$

$$v = 0$$

$\therefore$ The line is u-axis.

Case (v):

consider the half-plane $x > c_1$ in the z-plane.

$$x > c_1$$

$$\Rightarrow \frac{u}{u^2+v^2} > c_1$$

take reciprocal

$$\frac{u^2+v^2}{u} = u/c_1$$

$$u^2+v^2 = u/c_1$$

Adding $(1/2c_1)^2$ on both side

$$u^2+v^2+(1/2c_1)^2 < \frac{u}{c_1} + \left(\frac{1}{2c_1}\right)^2$$

$$u^2 - \frac{u}{c_1} + (1/2c_1)^2 + u^2 < \frac{1}{(2c_1)^2}$$

$$(u - 1/2c_1)^2 + v^2 < (1/2c_1)^2$$

(1.2) The half-plane $x > 0$ in the z-plane is mapped into the interior of the circle having the centre $(1/2c_1, 0)$ and radius $(1/2c_1)$, v-axis is the tangent to the circle.

2) Discuss the transformation $w = z^2$.

Soln:-

The transformation $w = z^2$ is a 1=1 Correspondence between the non-zero points of z-plane and w-plane.

The transformations is conformal at all points except $z=0$

$\therefore \frac{dw}{dz} = 2z$ and we have

$$w = z^2 \rightarrow ①$$

$$u+iv = (x+iy)^2$$

$$= x^2 - y^2 + i2xy$$

$$u+iv = x^2-y^2+i2xy$$

$$u = x^2-y^2 \rightarrow ②$$

$$v = 2xy \rightarrow ③$$

Case(i) :-

Let $x=k$ then $v=2ky \Rightarrow y = \frac{v}{2k}$

Then $u = k^2 - \frac{v^2}{4k^2}$

$$u = \frac{4k^4 - v^2}{4k^2}$$

$$4k^2 u = 4k^4 - v^2$$

$$v^2 = 4k^4 - 4k^2 u$$

$$v^2 = 4k^2 (k^2 - u)$$

whic is a parabola on the v-axis with focus $(k^2, 0)$

III^{ly} let $y = \lambda$

Then $v = 2x\lambda \Rightarrow \frac{v}{2\lambda} = x$

$$u = \frac{v^2}{4\lambda^2} - \lambda^2$$

$$u = \frac{u^2 - 4\lambda u}{4\lambda^2}$$

$$4\lambda^2 u = v^2 - 4\lambda^4$$

$$v^2 = 4\lambda^2 u + 4\lambda^4$$

$$v^2 = 4\lambda^2 (u + \lambda^2)$$

$\therefore$ which is a parabola on the v-axis with focus $(\lambda^2, 0)$

Hence the system of line $x = k$ and $y = \lambda$ is mapping into asystem of parabolas.

The two families of the parabolas are orthogonal.

Since the family of line $x = k$ and $y = \lambda$ is orthogonal in the z-plane.

case (ii) :-

The line is y-axis

(ie) $x = 0$ $\quad v = 2xy$

$\Rightarrow v = 0$

$\therefore$ The line y-axis is mapped into the u-axis

$||^{ly}$ The line is x-axis

(ie) $y = 0$ $\quad v = 2xy$

$\Rightarrow v = 0$

$\therefore$ The line x-axis is mapped into the u-axis.

case (iii) :-

The straight line $u = k$ and $v = \lambda$ then we have,

$$x^2 - y^2 = k \quad \Rightarrow \frac{x^2}{k} - \frac{y^2}{k} = 1$$

$$2xy = \lambda \qquad xy = k$$

(ie) The lines $u = k$ and $v = \lambda$ on the w-plane map into orthogonal families of rectangular hyperbola.

(ie) The two families of hyperbola have the same centre (0,0) and the asymptotes of one family of hyperbolas form the axies of other family of hyperbola.

Case (iv) :-

Let $z = re^{i\theta}$

$w = z^2$

$w = (re^{i\theta})^2$

$w = r^2 e^{2i\theta}$

where $r^2 = \lambda$

$\Rightarrow w = \lambda e^{2i\theta}$

Let $2\theta = \phi$

$\Rightarrow w = \lambda e^{i\theta}$

If z describes and arc of the a circle in the z-plane with radius r and angle θ then w describes a circle of radius r^2 and angle 2θ at the origin.

(ie) The mapping is doubled in its angle.

Case (v) :-

consider the circle at centre 'a' and radius 'c'

Then we have.

$|z-a| = c$

$z - a = ce^{i\theta}$

$z = ce^{i\theta} + a$

$w = z^2$

$w = (a + ce^{i\theta})^2$

$w = a^2 + c^2 e^{2i\theta} + 2ac e^{i\theta} \rightarrow (1)$

$= a^2 + ce^{i\theta}(ce^{i\theta} + 2a)$

from (1)

$w - a^2 = c^2 e^{2i\theta} + 2ac e^{i\theta}$

from ① :

$$w - a^2 = c^2 e^{2i\theta} + 2ace^{i\theta}$$

Adding c^2 on b.s

$$w - a^2 + c^2 = c^2 + c^2 e^{2i\theta} + 2ace^{i\theta}$$

$$w - a^2 + c^2 = c^2(1 + e^{2i\theta}) + 2ace^{i\theta}$$

$$= c^2(e^{i\theta}.e^{-i\theta} + e^{i\theta}.e^{i\theta}) + 2ace^{i\theta}$$

$$= c^2 e^{i\theta}(e^{-i\theta} + e^{i\theta}) + 2ace^{i\theta}$$

$$= 2ce^{i\theta}(c\cos\theta + a) \rightarrow ③$$

$\left[\because e^{i\theta}.e^{i\theta} = e^{0} = 1\right.$

$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

$\left.2\cos\theta = e^{i\theta} + e^{-i\theta}\right]$

Put $c = a$

$$w - a^2 + a^2 = 2ae^{i\theta}(a\cos\theta + a)$$

$$w = 2a^2 e^{i\theta}(1 + \cos\theta)$$

$$Re^{i\theta} = 2a^2 e^{i\theta}(1 + \cos\theta)$$

$$R = 2a^2(1 + \cos\theta)$$

which is a cartioid.

(2) Discuss the transsformation $w = e^z$

Soln :-

e^z is analytic in the all the region.

The transformation $w = e^z$ is a 1-1 corresspondance between the non-zero points of z-plane and w-plane.

$\therefore$ The transformation is conformal at all points.

$\frac{dw}{dz} = e^z$ and we have

$$w = e^z$$

$$u + iv = e^{x+iy}$$

$$= e^x . e^{iy} = e^x.[2\cos y + i\sin y]$$

$$u = e^x \cos y = e^x \cos y + ie^x \sin y$$

$\rightarrow ①$

$$v = e^x \sin y \rightarrow ②$$

squaring ① + ② $\Rightarrow u^2+v^2 = e^{2x}$

$$\frac{②}{①} \Rightarrow \frac{v}{u} = \frac{e^x \sin y}{e^x \cos y}$$

$$\frac{v}{u} = \tan y$$

case (i) :-

Let $x=c$ in z plane we get

$$u^2 + v^2 = e^{2c}$$

which is a circle in the w- plane with radius $e^{\sqrt{2c}}$

(ie) The line segment $x=c$ is transformed into concentric circle with centre at the origin and lies with in the radius $e^{\sqrt{2c}}$

case (ii) :-

when $x=0$

Then $u^2+v^2=1$

which is circle in the w-plane with

$\therefore$ The y axis in the z-plane is transformed into unit circle $u^2+v^2=1$.

case (iii) :-

when $y=k$ we have

$$\frac{v}{u} = \tan y$$

$$\frac{v}{u} = \tan k$$

$v = u \tan k$ where $\tan k = m$
$c=0$

$y = mx+c$ slope of the line

$\therefore$ which is a straight line

case (iv) :-

For particular cases $y=a$, $y=b$ then $b-a < 2\pi$ corresponts to a wedge in the w-plane bounded by the radial lines.

case (v) :-

when $y=0$

$\frac{v}{u} = \tan\theta$

$\frac{v}{u} = 0$

$v = 0$

$u^2+v^2 = e^{2x}$

Then $u^2 = e^{2x}$

$u^2 = e^x \cdot e^x$

$\Rightarrow u = e^x$

In w-plane $v=0$, $u>0$

Hence x-axis in z-plane is the formed into +ve u-axis in w-plane.

case (vi) :-

put $y=\pi/2$

$\frac{v}{u} = \tan y$

$\Rightarrow \frac{v}{u} = \tan \pi/2$

$\frac{v}{u} = \infty$

If $u=0$, we get ∞ so $u=0$

Then $v=e^x>0$

which is ~~the~~ +ve side of v-axis in w-plane.

case (vii) :-

$$y = \pi$$

$$\frac{v}{u} = \tan \pi$$

$$\frac{v}{u} = 0$$

$$① \Rightarrow u = e^{x} \cos y$$

$$= e^{x} \cos(\pi)$$

$$= -e^{x} < 0$$

$v = 0$, $u < 0$ which o-ve side of v-axis in w-plane

case (ix) :-

$$y = 2\pi \quad , \frac{v}{u} = \tan 2\pi = 0$$

$$v = 0 \Rightarrow u = e^{x} \cos 2\pi = e^{x} > 0$$

$$u > 0$$

which is +ve side of u-axis in w-plane is mapped onto the whole of the w-plane except the point $w = 0$ and $w = \infty$.

18|8 Discuss the inverse transformation $W = \sqrt{z}$ A

$w = \sqrt{z}$ $w^2 = z$

Given $W = \sqrt{z}$

$w = u+iv$

$z = x+iy$

Squaring on both side

$w^2 = z$

$(u+iv)^2 = (x+iy)$

$W^2 = z$

$u^2+2iuv-v^2 = x+iy$

where, $W = u+iv$, $z = x+iy$

$x = u^2-v^2$

$y = 2uv$

$(u+iv)^2 = x+iy$

$u^2-v^2+2iuv = x+iy$

Equating the real & imaginary part

$x = u^2-v^2$ $y = 2uv$

we consider the inverse transformation $z = w^2$

CASE: I

Let $v = c$ in w plane we have

$x = u^2-c^2$; $y = 2uc$

$\Rightarrow u = \frac{y}{2c}$

Sub in x

$x = u^2-v^2$

$x = \left(\frac{y}{2c}\right)^2 - c^2$

$= \frac{y^2}{4c^2} - c^2$

$x = \frac{y^2-4c^4}{4c^2}$

$\Rightarrow 4xc^2 = y^2-4c^4$

$4c^2x + 4c^4 = y^2$

$$y^2 = 4c^2\,[x + c^2]$$

$y^2 = 4ax$

which is a Parabola.

The infinite strip $0 \le v \le c$ goes onto the interior of a parabola.

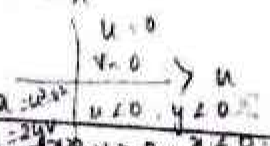
$$y^2 = 4c^2\,[x + c^2]$$

CASE :- ii

The first quadrant in W Plane namely $u \geq 0$, $v > 0$ is mapped to the region $x > 0$, $y > 0$ which is the upper half of z plane

iii

The 2^{nd} quadrent in the w plane $u < 0$, $v < 0$ is mapped to the region $x < 0$, $y < 0$ which is the lower half of z plane



Hence the upper half of ω plane goes to entire z plane

Discuss the transformation $w = e^z$

Given $w = e^z \Rightarrow w = e$

where, $w = u + iv$ $\quad z = x + iy$

$$u + iv = e^{x+iy}$$

$$u + iv = e^x \cdot e^{iy}$$

B

$= e^{x}.(\cos y + i\sin y)$

$u + iv = e^{x}\cos y + i\, e^{x}\sin y$

squading real & Im. P

$u = e^{x}\cos y \rightarrow (1)$ $\quad v = e^{x}\sin y \rightarrow (2)$

squaring & Adding (1) & (2)

$u^2+v^2 = e^{2x}\cdot 1 \rightarrow (3)$ $\quad u^2+v^2 = e^{2x}(\sin^2y+\cos^2y) = e^{2x}(1)$

$\frac{(2)}{(1)} \Rightarrow \frac{v}{u} = \tan y \rightarrow (4)$

CASE :- I $\quad x$ is constant

In z plane put $x = c$ we have

$u^2+v^2 = e^{2x}$

$(3) \Rightarrow u^2+v^2 = e^{2c}$

which is circle

ie, the line $\|^{el}$ to y axis In the z Plane is transformed in to concentric circle in centre at the origion in the w Plane.

The line segment $0 \le x \le c$ trans formed in concentric circle with centre at origin and lies with in the circle with the radius $e^{\sqrt{2c}}$

CASE: II

When $x=0$ we have.

(3) $\Rightarrow$ $u^2+v^2=1$

The y axis transformed into unit circle with centre origin in the w plane

CASE :- III

$$y = \text{constant}$$

Let $y=k$ we have

(4) $\Rightarrow$ $\frac{v}{u} = \tan k$

$y=mx+c$

$\Rightarrow$ $v = u \underbrace{\tan k}_{\text{slope}} + c$

~~$v=u\tan k$~~ which represent a st. Line $v=u\tan k+c$, where $c=0$ through the origin.

Hence the line ‖ to x axis in y axis z plane is transformed in to radial line in w plane.

line travel = x axis
radial line = y axis

c

CASE :- IV

For particular cases for $y=a$, $y=b$

Then $b-a$ less than 2π, corresponds to the wedge in the w plane bounded by radial lines.

intersect pt

a. e^{a} b

CASE :- V

④ ⇒ $\frac{v}{u} = \tan(0)$, $v=0$ ⑤ ⇒ $\boxed{v=0}$ ⇒ $u^2 = e^{2x}$.

When $y=0 \Rightarrow V=0$ and $u=\sqrt{e^{2x}}$

$u<0$

in w plane $V=0$, $u>0$.

Hence x axis in z plane is transformed in to +ve u axis in w plane.

CASE :- VI

put $y=\pi/2$ $y=\pi/2$

④ ⇒ $\frac{v}{u} = \tan\frac{\pi}{2}$ ⇒ $\tan\infty$ $\pi/4$

⇒ $u=0$

⇒ $V=\sqrt{e^{2x}}$ which is greather than zero

ie, which is +ve side of v axis in W plane

CASE :- VII

put $y=\pi \Rightarrow \frac{v}{u} = \tan\pi$

$\Rightarrow \frac{v}{u} = \frac{\sin 180^\circ}{\cos 180^\circ}$

$\frac{v}{u} = \frac{0}{-1} = 0 \Rightarrow v = 0$

① $\Rightarrow$ $u = e^x \cos y$

$u = e^x \cos \pi$

$u = -e^x$ which is less then zero

which is negative side of u axis in w plane.

CASE :- VIII

put $y = 3\pi/2$

④ $\Rightarrow \frac{v}{u} = \tan 3\pi/2$

$= \tan 270^\circ$

$= \tan (180^\circ + 90^\circ)$

$\frac{v}{u} = \infty$

$\Rightarrow \boxed{u = 0}$

and ② $\Rightarrow$ $v = e^x \sin 3\pi/2$

$= e^x (-1)$

$= -e^x$ is less then zero

which is negative side of v axis in w plane.

CASE :- IX. put $y = 2\pi$ then

④ $\rightarrow \frac{v}{u} = \tan 2\pi \Rightarrow \frac{v}{u} = 0 \Rightarrow v = 0$

and by ① $\Rightarrow u = e^x \cos 2\pi$

$= e^x (1)$

$= e^x > 0$

which is +ve side of in w plane.

---

18 | Discuss the transform $w = \sin z$

Given $w = \sin z$

where $w = u+iv$, $z = \sin (x+iy)$

$u+iv = \sin (x+iy)$ $\qquad \cos iy = \cosh y$, $\sin iy = i \sinh y$

$= \sin x \cos iy + \cos x \sin iy$

$u+iv = \sin x \cosh y + i \cos x \sinh y$

Equating real & imaginary part

$u = \sin x \cosh y$ ① ; $v = \cos x \sinh y$ ②

the real axis z plane is $y=0$ and

when $y=0$ from ① & ② we get

$u = \sin x$; $v = 0$

Since, here $\sin x$ takes values b/w -1 and 1,

the image of real axis, is the segment

$-1 \leq \sin x \leq 1$

$-1 \leq u \leq 1$

which is u axis.

III^y If $x=0$ we get from ① & ②

$u=0 \quad v=\sinh y$

$\therefore$ If $y>0$ then $\sinh y$ is +ve.

If $y<0$ then $\sinh y$ is negative.

Hence the upper of the image axis in the z plane maps into the upper half the imaginary axis of w plane, while lower halves of both correspondes with one an another.

Eliminating x from ① and ②

$$\frac{u^2}{\cosh^2 y}+\frac{v^2}{\sinh^2 y}=1$$

when $y=c$ we have

$$\frac{u^2}{\cosh^2 c}+\frac{v^2}{\sinh^2 c}=1 \rightarrow ③$$

three represent an ellipse with semi axis $(\cosh c, \sinh c)$.

The foci are $(\pm ae, 0)$

$$ae=\pm\sqrt{a^2-b^2}$$
$$=\pm\sqrt{\cosh^2 c-\sinh^2 c}$$
$$=\pm\sqrt{1}$$
$$\boxed{ae=\pm 1}$$

E

This $\overset{\text{co-ordinates}}{\text{quadrants}}$ are independent of c and hence the ellipse are confocal.

Thus, lines $\parallel^{el}$ to the real axis of the z plane map into confocal ellipse in the w plane.
and its ellipse diagram

Eliminating y from ① & ② we get

$$\frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = 1$$

when $x = k$ then

$$\frac{u^2}{\sin^2 k} - \frac{v^2}{\cos^2 k} = 1 \longrightarrow ④$$

which represent the system of hyperbola with foci $(\pm ae, 0)$.

where $ae = \pm\sqrt{a^2+b^2}$

$= \pm\sqrt{\sin^2 k + \cos^2 k}$

$= \pm\sqrt{1}$

$ae = \pm 1$

Hence the lines $\parallel^{el}$ to the imaginary axis of z plane maps into confocal hyperbola.

Problems:

Under the transformation $w = \frac{1}{z}$ find the image of the curve $|z-2i| = 2$.

Soln

Given $W = \frac{1}{z}$, $|z-2i| = 2$

$$W = \frac{1}{z}$$

where $u+iv = W$, $z = x+iy$

$$u+iv = \frac{1}{x+iy}$$

$$= \frac{1}{x+iy} \times \frac{x-iy}{x-iy}.$$

$$u+iv = \frac{x-iy}{x^2+y^2}$$

$$u = \frac{x}{x^2+y^2} \rightarrow (1) \qquad v = \frac{-y}{x^2+y^2} \rightarrow (2)$$

Given $|z-2i| = 2$

$$|x+iy-2i| = 2$$

$$|x+i(y-2)| = 2$$

$$x^2+(y-2)^2 = 4$$

$$x^2+y^2+4-4y = 4$$

$$x^2+y^2-4y = 0 \rightarrow (3)$$

which is the sqation of circle in z plane with centre $(0,2)$ and radious 2

$$x^2+y^2 = \frac{1}{u^2+v^2} \text{ from (1) \& (2)}$$

$$x = \frac{u}{u^2+v^2} \qquad y = \frac{-v}{x^2+y^2}$$

F

③ $\Rightarrow x^2+y^2-4y=0$

$$\frac{u^2}{(u^2+v^2)^2}+\frac{v^2}{(u^2+v^2)^2}+4\left(\frac{v}{(u^2+v^2)}\right)=0$$

$$\frac{u^2+v^2+4v\,(u^2+v^2)}{(u^2+v^2)^2}=0.$$

$$u^2+v^2+4v(u^2+v^2)=0$$

$\div\ (u^2+v^2) \Rightarrow 1+4v=0$

which is eqns of st line in w plane.

FIND the image of the strip $1/4 < y < 1/2$ under $w=\frac{1}{z}$.

$<$ negative
$>$ positive

soln

Given $w=1/z$

we know that $x=\frac{u}{u^2+v^2}$ → ① ; $y=\frac{-v}{u^2+v^2}$ → ②

consider, $y>\frac{1}{4}$

② $\Rightarrow \frac{-v}{u^2+v^2}>\frac{1}{4}$

$-4v>u^2+v^2.$

$\Rightarrow 0>u^2+v^2+4v$

$\Rightarrow u^2+v^2+4v<0$

which is circle in the negative axis

and

$$y > \frac{1}{4}$$

$$\frac{-v}{u^2+v^2} > \frac{1}{4}$$

$$-4v > u^2+v^2$$

$$\Rightarrow -4v+4 > u^2+v^2+4$$

$$\Rightarrow 4 > u^2+v^2+4v+4$$

$$\Rightarrow 4 > u^2+(v+2)^2$$

$$\Rightarrow u^2+(v+2)^2 < 4$$

which is on the circle on the real axis.

consider, $y < 1/2$

$$\frac{-v}{u^2+v^2} < 1/2$$

$$-2v < u^2+v^2$$

$$\Rightarrow -2v+1 < u^2+v^2+1$$

$$\Rightarrow 1 < u^2+v^2+2v+1$$

$$\Rightarrow 1 < u^2+(v+1)^2$$

which is the circle in the positive axis.

Determine the angle of rotation of ~~ ~~

transformation $W = z^2$ at $1+i$.

Soln

Given $W = z^2$

where $u = x+iy$ $z = x+iy$

$$u+iv = (x+iy)^2$$

$$u+iv = x^2 - y^2 + 2ixy$$

$$u = x^2 - y^2 \rightarrow ① \quad ; \quad v = 2xy \rightarrow ②$$

CASE :-(i)

at (1,1) we have from ① & ②

$u = x^2 - y^2$, $u = (1-1) = 0$

$u = 0$ and $v = 2$

$\therefore$ $u = x^2 - y^2$: $v = 2xy$

$0 = x^2 - y^2$ $\quad$ $2 = 2xy$

$x^2 = y^2$ origin same $\quad$ $xy = 1$ similar

$\boxed{x = y}$ $\quad$ $\boxed{y = 1/x}$

ie, the half lines $y = x$ and $y = 1/x$

intersect at the origion.

CASE : ii

let $x = 1$ from ① & ②

$u = 1 - y^2$; $v = 2y$

$y^2 = 1 - u$

$v = 2y$

$v^2 = 4y^2$

$y^2 = v^2/4$

$y^2 = \left(\frac{1}{4}\right) v^2$

$y^2 = 4a\,x$

which is a parabola.

| CASE :- III

Let $y = 1$ from ① & ②

$u = x^2 - 1 \quad v = 2x$

$u + 1 = x^2$.

$4u + 4 = 4x^2$.

$x^2 = \frac{1}{4}(4u + 4)$

BILINEAR TRANSFORMATION :

22/8

A transformation of the form.

$$w = \frac{az + b}{cz + d}$$

where, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$, a, b, c, d are complex no, is called B.L.T.

NOTE :-

* This transformation is also called as Mobius transformation
* This transformation is linear in as well as z.

(ie) $\omega = \dfrac{az+d}{cz+d}$.

$$\omega(cz+d) = az+b$$

$$\omega cz + \omega d = az+b$$

$$\omega d - b = az - \omega cz$$

$$\omega d - b = z(a-\omega c)$$

$$\boxed{z = \frac{\omega d - b}{a - \omega c}}$$

* If $d \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$ then

[illegible]

$ad - bc = 0$ and $z_1 \neq z_2$

Let $\omega_1 = \dfrac{az_1+b}{cz_1+d}$; $\omega_2 = \dfrac{az_2+b}{cz_2+d}$

$$\omega_1 - \omega_2 = \frac{az_1+b}{cz_1+d} - \frac{az_2+b}{cz_2+d}$$

$$= \frac{(az_1+b)(cz_2+d) - (az_2+b)(cz_1+d)}{(cz_1+d)(cz_2+d)}$$

$$= \frac{\cancel{acz_1z_2} + adz_1 + bcz_2 + bd - \cancel{acz_1z_2} - adz_2 - bcz_1 - bd}{(cz_1+d)(cz_2+d)}$$

$$\Rightarrow \frac{adz_1 + bcz_2 - adz_2 - bcz_1}{(cz_1+d)(cz_2+d)} = 0$$

$$\Rightarrow \frac{ad(z_1-z_2) - bc(z_1-z_2)}{(cz_1+d)(cz_2+d)} = 0$$

$$\Rightarrow ad(z_1-z_2) - bc(z_1-z_2) = 0$$

$$\Rightarrow (ad-bc)(z_1-z_2) = 0$$

ie, If $ad-bc=0$, the entire plane goes to a single ω plane which is trivial.

ie, $z_1 = z_2 = 0$

$$\Rightarrow z = 0, \; \omega = \frac{az+b}{cz+d} \; \omega = b/d$$

when $z_1 \neq z_2$ is solution is non trivial.

PARTICULAR CASES :-

(i) If $c=0$ and $a/d = 1$ then

$$\omega = \frac{az+b}{cz+d}$$

$$\omega = \frac{az}{cz+d} + \frac{b}{cz+d}$$

$$= \frac{az}{d} + b/d$$

$$w = z + b/d \Rightarrow \boxed{w = z + c}$$

which is a translational

ii) If $c = 0$ $b = 0$

$$w = \frac{az + b}{cz + d}$$

$$w = \frac{az}{cz+d} + \frac{b}{cz+d}$$

$$w = \frac{az}{d}$$

which is rotation.

iii) If $c = 0$

$$w = \frac{az + b}{cz + d}$$

$$w = \frac{az}{d} + \frac{b}{d} \Rightarrow \boxed{w = Bz + c}$$

which is translation and rotation

iv) If $a = 0$, $b = c$, $d = 0$

$$w = \frac{az + b}{cz + d}$$

$$= \frac{c}{cz} = 1/z$$

is Inversion.

THEOREM :

UNDER BLT known two pts in z Plane go to the same pt in ω Plane.

Soln

If $ad - bc \neq 0$ then $z_1 \neq z_2$

TO PROVE :- $\omega_1 \neq \omega_2$

Let $\omega_1 = \dfrac{az_1+b}{cz_1+d}$; $\omega_2 = \dfrac{az_2+b}{cz_2+d}$

If possible :

Let $\omega_1 = \omega_2$ Then $\omega_1 - \omega_2 = 0$

$$\omega_1 - \omega_2 = \frac{az_1+b}{cz_1+d} - \frac{az_2+b}{cz_2+d}$$

$$= \frac{(az_1+b)(cz_2+d) - (az_2+b)(cz_1+d)}{(cz_1+d)(cz_2+d)}$$

$$= \frac{acz_1z_2 + adz_1 + bcz_2 - acz_1z_2 - adz_2 - bcz_1}{(cz_1+d)(cz_2+d)}$$

$$\Rightarrow \frac{adz_1 + bcz_2 - adz_2 - bcz_1}{(cz_1+d)(cz_2+d)}$$

$$\Rightarrow ad(z_1 - z_2) - bc(z_1 - z_2) = 0$$

$\Rightarrow (ad-bc)(z_1-z_2)=0$

$\Rightarrow$ Either $(ad-bc)=0$ (or) $z_1-z_2=0$

But $ad-bc \neq 0$ and $z_1 \neq z_2$

which is a contradicat.

$\therefore \omega_1-\omega_2 \neq 0$

$\Rightarrow \omega_1 \neq \omega_2$

Now, two pts z plane will be in the same pt.

NOTE :-

If $\omega = \dfrac{az+b}{cz+d}$ is zero (or) infinity (∞)

* If $cz+d=0 \Rightarrow \omega=\infty$ and $z=-b/d$

* If $z=\infty \Rightarrow \omega = \dfrac{az+b}{cz+d}$

$\Rightarrow \omega = \dfrac{z(a+b/z)}{z(c+d/z)}$

$\omega = a/c$ //

FIXED POINTS :- (root) is f.p :-

points which are invariant under a transformation is called fixed pt (or) constant pt (or) invariant pt of transformation.

The fixed pt of $w = \frac{az+b}{cz+d}$

let $w = z \Rightarrow z = \frac{az+b}{cz+d}$

$\Rightarrow cz^2 - dz - az - b = 0$

$\Rightarrow cz^2 + (d-a)z - b = 0$

$\therefore a = c, \quad b = d-a, \quad c = -b$

$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-(d-a) \pm \sqrt{(d-a)^2 + 4bc}}{2c} \longrightarrow ①$

$= \frac{-(d-a) \pm \sqrt{d^2 + a^2 - 2da + 4}}{2c}$

CASE :- I

when $c = 0$ from ① $\Rightarrow z = \infty$

$\therefore$ The transformation becomes,

$z = \frac{az+b}{cz+d}$

$= \frac{\cancel{z}(a+b/z)}{z(c+d/z)}$

$z = \frac{a+b/z}{c+d/z}$ (or)

from ① $\Rightarrow \overset{\boxed{c=0}}{z} = \frac{az+b}{d}$

K

$$dz = az + b$$

$$\frac{(d-a)z}{b} = 1$$

$$\frac{(d-a)}{b} = 1/z.$$

$$z = \frac{b}{(d-a)}$$

CASE :- ii) $b=0$

$\therefore$ The transformation becomes,

$$z = \frac{az}{cz+d}$$

$$(cz+d) = az$$

$$cz^2+dz-az=0$$

$$cz^2+[d-a]z=0$$

$$z[cz+(d-a)]=0$$

$$\Rightarrow z=0 \quad , \quad z = \frac{a-d}{c}$$

$$① \Rightarrow z=0$$

$$z = \frac{2(a-d)}{2c} \quad \frac{(a-d) \pm (d-a)}{2c.}$$

CASE(iii) :-

$$(d-a)^2+4bc=0$$

$\therefore$ The transformation becomes.

$$① \Rightarrow z = \frac{a-d}{2c.}$$

coincident pts.

NOTE :-

CASE	Fixed pt
* $W = z+b$ (translation)	$z = \infty$
* $W = Bz$	$z = 0, z = \infty$
* $W = 1/z$	$z = \pm 1$

THEOREM :- 2

The set of all BLT is a group under composition.

proof :- consider $w = T(z) = \dfrac{az+b}{cz+d} \to ①$

let $z = \dfrac{z_1}{z_2}$ & $w = \dfrac{w_1}{w_2}$

$$\text{from } ① \Rightarrow \frac{w_1}{w_2} = \frac{a\left(\frac{z_1}{z_2}\right) + b}{c\left(\frac{z_1}{z_2}\right) + d}$$

$$= \frac{\frac{az_1 + bz_2}{\cancel{z_2}}}{\frac{cz_1 + dz_2}{\cancel{z_2}}}$$

$$\frac{w_1}{w_2} = \frac{az_1 + bz_2}{cz_1 + dz_2}.$$

In matrix format we can write

as, $\dfrac{w_1}{w_2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$

L

Since, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$ we have

$m_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a non sigolar of

the transformation.

Let $M_1 = a_1, b_1, c_1, d_1$ $M_2 = a_2, b_2, c_2 d_2$

be the matrices corresponding to the

transformation T_1, T_2 respectively.

Then, the composition of transformation

T_1, T_2 is given by

$W = T_2 T_1 (z)$

which is equal to

$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = M_2 M_1 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$

is also a BLT.

Hence the set is closure under

composition.

and w.r.T, composition is associative

since, matrix multiplication is

associative

The identity element under the operation is the transformation whose matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Also the inverse of transformation with matrix m is the transformation with $\frac{1}{m}$.

Hence the set of transformation under composition forms a group.

NOTE:-

since matrix multiplication is not commutative. the above group is non-abelian group under composition.

2418 THEOREM :- 3

The BLT which transforms $z_1 z_2 z_3$ into $w_1 w_2 w_3$ is $\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$

proof:-

Let $w = \frac{az+b}{cz+d}$

$\therefore w_1 = \frac{az_1+b}{cz_1+d}$; $w_2 = \frac{az_2+b}{cz_2+d}$; $w_3 = \frac{az_3+b}{cz_3+d}$

$$w - w_1 = \frac{az+b}{cz+d} - \frac{az_1+b}{cz_1+d}$$

M

$$(\omega-\omega_1)(cz+d)(cz_1+d) = aczz_1 + adz + bcz_1 + bd - aczz_1 - adz_1 - bcz - bd$$

$$= adz - bcz + bcz_1 - adz_1$$

$$= z(ad-bc) - z_1(ad-bc)$$

$$= (z-z_1)(ad-bc) \longrightarrow ①$$

III^ly

$$(\omega_2-\omega_3)(cz_2+d)(cz_3+d) = (z_2-z_3)(ad-bc) \longrightarrow ②$$

$$(\omega-\omega_3)(cz+d)(cz_3+d) = (z-z_3)(ad-bc) \longrightarrow ③$$

$$(\omega_2-\omega_1)(cz_2+d)(cz_1+d) = (z_2-z_1)(ad-bc) \longrightarrow ④$$

$$\frac{①\times②}{③\times④} \Rightarrow \frac{(\omega-\omega_1)(\omega_2-\omega_3)}{(\omega-\omega_3)(\omega_2-\omega_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Hence It is proved.

NOTE:-

suppose $z_3 = \infty$ R.H.S becomes

$$\text{R.H.S} \Rightarrow \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\Rightarrow \frac{z_3(z-z_1)\left(\frac{z_2}{z_3}-1\right)}{z_3\left(\frac{z}{z_3}-1\right)(z_2-z_1)}$$

$$\text{as } \lim_{z_3\to\infty} \frac{(z-z_1)(-1)}{(-1)(z_2-z_1)}$$

$$\therefore \frac{z-z_1}{z_2-z_1}$$

CROSS RATIO :-

Given four points z_1, z_2, z_3, z_4 taken in order. the ratio $\dfrac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}$ is called the cross ratio.

THEOREM :- 1

The cross ratio is preserve by a BLT

Pf :-

let $w = \dfrac{az+b}{cz+d}$ be the BLT

let the given pts be z_1, z_2, z_3, z_4 as in previous theorem.

$(w_1-w_3)(cz_1+d)(cz_3+d) = (z_1-z_3)(ad-bc) \rightarrow ①$

$(w_2-w_4)(cz_2+d)(cz_4+d) = (z_2-z_4)(ad-bc) \rightarrow ②$

$(w_1-w_4)(cz_1+d)(cz_4+d) = (z_1-z_4)(ad-bc) \rightarrow ③$

$(w_2-w_3)(cz_2+d)(cz_3+d) = (z_2-z_3)(ad-bc) \rightarrow ④$

① × ② ÷ ③ × ④ =>

$$\frac{(w_1-w_3)(w_2-w_4)}{(w_1-w_4)(w_2-w_3)} = \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}$$

THEOREM :- 2

BLT maps st. lines and circle.

proof :-

A BLT $w = \dfrac{az+b}{cz+d}$ is composite of elementary transformation namely translation, rotation, homothetic transformation and inversion.

CASE :- (i)

let $c = 0$ when $w = \dfrac{az+b}{cz+d}$

$$W = \frac{az}{d} + \frac{b}{d}$$

let $W_1 = \dfrac{a}{d} z \Rightarrow w = w_1 + b/d$

Here the transformation is a rotation and homothetic transformation followed by translation.

CASE :- (ii)

let $c \neq 0$

$$w = \frac{bc-ad}{c^2} \cdot \frac{1}{z+d/c} + a/c$$

$$R.H.S = \frac{bc-ad}{c^2} \cdot \frac{1}{z+d/c} + a/c$$

$$= \frac{bc-ad}{c^2} \cdot \frac{1}{\frac{zc+d}{c}} + a/c$$

$$= \frac{bc-ad}{c^2} \cdot \frac{c}{zc+d} + a/c$$

$$= \frac{bc-ad + a(zc+d)}{c(zc+d)}$$

$$= \frac{bc-ad+acz+ad}{c(zc+d)}$$

$$= \frac{c\,(b+az)}{c\,(zc+d)}$$

$$w = \frac{az+b}{cz+d}$$

It can be written as.

$$W_1 = z + d/c$$

$$\omega_2 = \frac{1}{\omega_1} \quad ; \quad \omega_3 = \frac{bc-ad}{c^2} \cdot \omega_2$$

$$\omega = \omega_3 + \frac{a}{c}$$

Hence the BLT is composed of

i) translation

ii) Inversion

iii) rotation

followed by homothetic transformation

iv) translation

These transformation except inversion carry st.lines into st.lines and circles into circles.

we now S.T the inversion $\omega = \frac{1}{z}$ carries st.lines and circles in to st. lines and circles.

consider the eqs of circle in z Plane.

$$\alpha(x^2+y^2)+2gx+2fy+\lambda=0$$

where α, g, f, λ are real. This represents a circle $\alpha \neq 0$ and a st.line.

when $\alpha = 0$

The inversion is given by

let $W = \frac{1}{z}$

where, $W = u+iv \quad z = x+iy$

$$u+iv = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$$

$$u = \frac{x}{x^2+y^2} \;;\; v = \frac{-y}{x^2+y^2}$$

$$\left.\begin{aligned} u^2+v^2 &= \frac{(x^2+y^2)}{(x^2+y^2)^2} = \frac{1}{x^2+y^2} \\ x &= \frac{u}{u^2+v^2} \quad y = \frac{-v}{u^2+v^2} \end{aligned}\right\} \rightarrow (2)$$

Sub (2) in (1)

$$\alpha \frac{1}{u^2+v^2} + 2g \frac{u}{u^2+v^2} - 2f \frac{v}{u^2+v^2} + \lambda = 0$$

$$\alpha + 2gu - 2fv + \lambda(u^2+v^2) = 0$$

This image is a circle when $\lambda \neq 0$ and is a st. line when $\lambda = 0$

Hence It is proved.

18 THEOREM:

If p, q are fixed pts under a BLT $w = \frac{az+b}{cz+d}$ then the transformation can be expressed in the form

i) $\frac{w-p}{w-q} = k.\frac{(z-p)}{(z-q)}$.

ii) If p is a co-incident fixed pt. then,

$$\frac{1}{w-p} = k + \frac{1}{z-p}$$

Proof :-

[i] let $W = \frac{az+b}{cz+d}$ $\therefore W_1 = \frac{az_1+b}{cz_1+d}$

$(w-w_1)(cz+d)(cz_1+d) = (z-z_1)(ad-bc)$

given fixed pts p, q

$\Rightarrow (w-p)(cz+d)(cp+d) = (z-p)(ad-bc) \rightarrow ①$

$(w-q)(cz+d)(cq+d) = (z-q)(ad-bc) \rightarrow ②$

$① \div ② \Rightarrow \frac{(w-p)}{(w-q)} \cdot \frac{(cp+d)}{(cq+d)} = \frac{z-p}{z-q}$

$\Rightarrow \frac{(w-p)}{(w-q)} \cdot k = \frac{(z-p)}{(z-q)}$

$\Rightarrow \frac{(w-p)}{(w-q)} = \frac{1}{k} \cdot \frac{(z-p)}{(z-q)}$

(or)

$\frac{(w-p)}{(w-q)} = k \cdot \frac{(z-p)}{(z-q)}$

[ii] If p is a co-incident pts of $(z-p)^2 = 0$

Now, $W = \frac{az+b}{cz+d}$.

If will given,

$(\ )wz + (\ )w + (\ )z + (\ ) = 0$

It should be reduced to $(z-p)^2 = 0$

The form should be,

$k(\omega - p)(z-p) + (\omega - z) = 0$

$k(\omega - p)(z-p) + (\omega - p - \overline{z-p}) = 0$

$k(\omega - p)(z-p) + (\omega - p) - (z-p) = 0$

$\div (\omega-p)(z-p) \Rightarrow$

$\Rightarrow k + \frac{1}{(z-p)} - \frac{1}{(\omega-p)} = 0$

$\Rightarrow \frac{1}{(\omega-p)} = k + \frac{1}{(z-p)}$

Hence It is proved

SPECIAL BLT:

*upper half plane imaginary $z \geq 0$ onto the unit circular disc $|w| \leq 1$.

*Right half plane real part of $z \geq 0$ onto the unit circular disc $|w| \leq 1$.

*unit circular disc $|z| \leq 1$ onto the unit circular disc $|w| \leq 1$.

PROBLEM:

The BLT which maps imaginary $z \geq 0$ onto $|w| \leq 1$ are of the form $w = e^{i\lambda} \cdot \frac{z - z_1}{z - \bar{z}_1}$, where λ is real.

Proof:

Let $w = \frac{az+b}{cz+d} \longrightarrow ①$ with imaginary

$z=0$ get transformation to unit circle $|W|=1$.

consider,

$z=0,\ z=1,\ z=\infty$ on the real axis

$$W=\frac{b}{d}\ ,\ W=\frac{a+b}{c+d}\ ,\ W=\frac{\cancel{z}(a+b/z)}{\cancel{z}(c+d/z)}=\frac{a}{c}$$

All these pts are going to $|W|=1$.

$$\left|\frac{b}{d}\right|=1 \quad \text{if } z=0 \qquad \longrightarrow (*)$$

$$\left|\frac{a}{c}\right|=1 \quad \text{if } z=\infty$$

$$\frac{a}{c}=e^{i\lambda} \text{ where } \lambda \text{ is real}$$

P.T $|W|=1 \Rightarrow 1=e^{i\lambda}\frac{(z-z_1)}{z-z_2}$

$(z-z_2)=e^{i\lambda}(z-z_1)$

$k=e^{i\lambda}t$

$z=re^{i\theta} \Rightarrow$ complex no

$$(1) \Rightarrow \omega=\frac{a\,(z+b/a)}{c\,(z+d/c)}$$

$$=e^{i\lambda}\frac{(z-z_1)}{(z-z_2)} \longrightarrow (I)$$

where $z_1=-b/a$, $z_2=-d/c$

$$\left|\frac{z_1}{z_2}\right|=\left|\frac{b}{a}\cdot\frac{c}{d}\right|=\left|\frac{b}{a}\right|\left|\frac{c}{d}\right|$$

$$\left|\frac{z_1}{z_2}\right|=1 \text{ (or) } |z_1|=|z_2| \longrightarrow (II)$$

$\therefore z=1$ also goes to $|W|=1$

$$(I) \Rightarrow 1=\left|e^{i\lambda}\frac{(1-z_1)}{(1-z_2)}\right|$$

$$|1-z_2| = |e^{i\lambda}| \cdot |1-z_1| \quad , \quad |e^{i\lambda}| = 1$$

$$|1-z_2| = |1-z_1|$$

$$|1-z_2|^2 = |1-z_1|^2$$

$$(1-z_2)(\overline{1-z_2}) = (1-z_1)(\overline{1-z_2})$$

$$(\cancel{1} - \bar{z}_2 - z_2 + z_2\bar{z}_2) = (\cancel{1} - \bar{z}_1 - z_1 + z_1\bar{z}_1)$$

$$-(z_2+\bar{z}_2) = -(z_1+\bar{z}_1)$$

$$z_2+\bar{z}_2 = z_1+\bar{z}_1$$

$$2 \text{ R.P of } z_2 = 2 \text{ R.P of } z_1$$

(II) $\Rightarrow |z_1| = |z_2|$

$$z_1 = z_2 \; ; \; \bar{z}_2 = \bar{z}_1 \text{ (or) } z_2 = \bar{z}_1$$

(I) becomes,

$$w = e^{i\lambda} \frac{(z-z_1)}{(z-z_2)}$$

$$= e^{i\lambda} \frac{(z-z_1)}{(z-\bar{z}_2)}$$

$(w_1-w_3)\ (w_2$

$$w = e^{i\lambda} \frac{(z-z_1)}{(z-\bar{z}_1)}$$

Hence it is proved.

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}$$

Problems : 1

Find the BLT which maps pt $z_1=0$, $z_2=-i$, $z_3=-1$ into $w_1=i$, $w_2=1$, $w_3=0$

Soln

Let $w = \frac{az+b}{cz+d}$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-i)(1-0)}{(w-0)(1-i)} = \frac{(z-0)(-i+1)}{(z+1)(-i-0)}$$

$$\frac{(w-i)}{w(1-i)} = \frac{z(-i+1)}{(z+1)(-i)}$$

$$\frac{(w-i)}{(w-wi)} = \frac{(z-zi)}{(-zi-i)}$$

$$(w-i)(-zi-i) = (w-wi)(z-zi)$$

$$-wzi-wi+z-1 = wz-wzi-wzi-wz$$

$$-wzi-wi+z-1 = -2wzi$$

$$-wi+z-1 = -wzi$$

$$-wi+wzi+z-1 = 0$$

$$-wi+wzi = (1-z)$$

$$-w(1-z)i = (1+z)$$

$$wi = \frac{(1+z)}{(1-z)}$$

$$w = \frac{1}{i}\left[\frac{(1+z)}{(1-z)}\right]$$

problem : 2

Find the cross ratio for the BLT

$(i, 2, 0, 1)$

R

Given $z_1 = i,\ z_2 = 2,\ z_3 = 0,\ z_4 = 1$

$$\text{Cross ratio} = \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}$$

$$= \frac{(i-0)(2-1)}{(i-1)(2-0)}$$

$$= \frac{i(1)}{(i-1)(2)}$$

$$= \frac{i}{2i-2}$$

$$= \frac{1}{2} - \frac{i}{2}.$$

Problem :- 3

FIND the BLT which the from $w_1 = 1$ $w_2 = i$ $w_3 = -1$ on to $z_1 = 2,\ z_2 = i,\ z_3 = -2.$

Soln

Given $w_1 = 1$ $w_2 = i$, $w_3 = -1$, $z_1 = 2, z_2 = i, z_3 = -2$

into

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-1)(i+1)}{(w+1)(i-1)} = \frac{(z-2)(i+2)}{(z+2)(i-2)}$$

$$\frac{(wi+w-i-1)}{(wi-w+i-1)} = \frac{(zi+2z-2i-4)}{(zi-2z+2i-4)}$$

$$(wi+w-i-1)(zi-2z+2i-4) = (zi+2z-2i-4)(wi-w+i-1)$$

$-2w \quad -4w$

$$-wz - wz2i \boxed{-wz} - 4wi + wzi \boxed{-2wz} + 2wi - 4w + z + 2zi + 2 + 4i - 2i + 2z + 2i + 4 = -wz - wzi - z - zi + 2wzi - 2wz + 2zi - 2z + 2w - 2wi + 2 + 2i - 4wi + 4w - 4i + 4$$

Soln

Given $z_1 = i,\ z_2 = 2,\ z_3 = 0,\ z_4 = 1$

$$\text{Cross ratio} = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

$$= \frac{(i-0)(2-1)}{(i-1)(2-0)}$$

$$= \frac{i(1)}{(i-1)(2)}$$

$$= \frac{i}{2i-2}$$

$$= \frac{1}{2} - \frac{i}{2}.$$

Problem :- 3

FIND the BLT which the from $w_1 = 1\ w_2 = i$

$w_3 = -1$ on to $z_1 = 2,\ z_2 = i,\ z_3 = -2.$

Soln

Given $w_1 = 1\ \ w_2 = i,\ w_3 = -1,\ z_1 = 2,\ z_2 = i,\ z_3 = -2$

into

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

$$\frac{(w-1)(i+1)}{(w+1)(i-1)} = \frac{(z-2)(i+2)}{(z+2)(i-2)}$$

$$\frac{(wi + w - i - 1)}{(wi - w + i - 1)} = \frac{(zi + 2z - 2i - 4)}{(zi - 2z + 2i - 4)}$$

$$(wi + w - i - 1)(zi - 2z + 2i - 4) = (zi + 2z - 2i - 4)(wi - w + i - 1)$$

$-2w\ \ -4w$

$-wz - wz2i\ \boxed{-wz} - 4wi + wzi\ \boxed{-2wz} + 2wi - 4w + z + 2zi$

$+ 2 + 4i - 2i + 2z + 2i + 4 = -wz - wzi - z - zi + 2wzi$

$-2wz + 2zi - 2z + 2w - 2wi + 2 + 2i - 4wi + 4w - 4i + 4$

$$-wzi - 3wz - 6w - 2wi + zi + 3z + 6i + 6 = -3wz + wzi - 3z + zi + 6w - 6wi - 2i + 6$$

$$-wzi - \cancel{3wz} - 6w - 2wi + \cancel{zi} + 3z + 6i + \cancel{6} + \cancel{3wz} - wzi + 3z - \cancel{zi} - 6w + 6wi + 2i - \cancel{6} = 0$$

$$-2wzi - 12w + 4wi + 6z + 8i = 0$$

$$\div 2 \Rightarrow wzi - 6w + 2wi + 3z + 4i = 0.$$

$$\cancel{-6w + 2wi + 4i = -wzi - 2wi}$$

$$\cancel{2w(2i(w+2) = +wi(-z-1)} - 6w$$

$$-6w + wzi + 2wi + 3z + 4i = 0$$

$$w(zi + i - 6) = -3z - 4i$$

$$w = \frac{(-3z - 4i)}{(zi + i - 6)}$$

on to:-

$$wzi - 6w + 2wi + 3z + 4i = 0$$

$$wzi + 3z = 2wi + 4i - 6w$$

$$z(wi + 3) = 2wi + 4i - 6w$$

$$\therefore z = \frac{2wi + 4i - 6w}{(wi + 3)}$$

problem :- 4

Find the value of K, of two pt fixed pt are given by (2, 3)

soln

Let $w = \dfrac{az+b}{cz+d}$ $\quad w_1 = \dfrac{az_1+b}{cz_1+d}$

$(w-w_1)(cz+d)(cz_1+d) = (z-z_1)(ad-bc)$

given the fixed pt $(2,3)$

$\Rightarrow (w-2)(cz+d)(2c+d) = (z-2)(ad-bc)$

$|||^{ly}$ $(w-3)(cz+d)(3c+d) = (z-3)(ad-bc)$

① ÷ ② $\Rightarrow$

$$\frac{(w-2)}{(w-3)}\frac{(2c+d)}{(3c+d)} = \frac{(z-2)}{(z-3)}$$

$$\frac{(w-2)}{(w-3)}\cdot k = \frac{(z-2)}{(z-3)}$$

$$k = \frac{(z-2)}{(z-3)}\cdot\frac{(w-3)}{(w-2)}$$

problem :5

$\frac{(w_1-w_3)(w_2-w_4)}{(w_1-w_4)(w_2-w_3)}$

check the whether Cross ratio preserves

BLT $z_1=0$, $z_2=-i$, $z_3=-1$, $z_4=2$ and $w_1=2i$

$w_2=i$, $w_3=1$, $w_4=0$

$\frac{(w_1-w_3)(w_2-w_4)}{(w_1-w_4)(w_2-w_3)}$

soln

$$\frac{(w_1-w_3)(w_2-w_4)}{(w_1-w_4)(w_2-w_3)} = \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}$$

$$\frac{(2i-1)(i-0)}{(2i-0)(i-1)} = \frac{(0+1)(-i+2)}{(0-2)(-i+1)}$$

$$\frac{(2i-1)(i)}{(2i)(i-1)} = \frac{(1)(-i+2)}{(-2)(-i+1)}$$

$$\frac{(-2-i)}{(-2-2i)} = \frac{(-i+2)}{+(2i-2)}$$

T

$(-2-i)(2i-2) = (2-i)\cdot(-2-2i)$

$-4i+4+2+2i = -4-4i+2i+2$

$-\cancel{4i}+4+2+\cancel{2i}+4+\cancel{4i}-\cancel{2i}-\cancel{2}=0$

$+4i+4 \neq 0 \Rightarrow 8=0$

Cross. r = 8

From the BLT $\frac{3z+4}{5z-1}$ find value of k

with $(5,4)$

Given $\frac{3z+4}{5z-1}$ the pt $(5,4)$

$$= \frac{3(5)+4}{5(4)-1}$$

$$= \frac{15+4}{20-1}$$

$$= \frac{19}{19}$$

$$k = 1$$ //.

UNIT - V

Definite Integrals

Defn:

Let $w(t) = u(t) + iv(t)$ be a Continuous Complex valued function defined on the interval $[a,b]$ then the value of

$$\int_a^b w(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

Eg: (1)

Solve the equations of Complex Valued functions $e^{i6t}, 0 \le t \le \frac{\pi}{2}$

Solutions:

The Definite Integral of the Complex function

is $\int_a^b w(t)dt$

(ie)
$$\int_0^{\pi/2} e^{i6t} dt = \int_0^{\pi/2} (\cos 6t + i\sin 6t)dt$$

$$= \int_0^{\pi/2} \cos 6t + i\int_0^{\pi/2} \sin 6t\, dt$$

$$= \left.\frac{\sin 6t}{6}\right]_0^{\pi/2} + i\left[-\frac{\cos 6t}{6}\right]_0^{\pi/2}$$

$$= \frac{1}{6}\left[0 - 0 - i(-1-1)\right]$$

$$= \frac{1}{6}[2i] = \frac{i}{3}$$

Eg: (2)

Find $\int_0^{\pi/6} e^{i2t} dt$

Solution:

Given
$$\int_0^{\pi/2} e^{i2t} dt = \int_0^{\pi/6} (\cos 2t + i\sin 2t)dt$$

$$= \int_0^{\pi/6} \cos 2t\, dt + i\int_0^{\pi/6} \sin 2t\, dt$$

$$= \left.\frac{\sin 2t}{2}\right]_0^{\pi/6} + i\left[-\frac{\cos 2t}{2}\right]_0^{\pi/6}$$

$$= \frac{1}{2}\left[\sin\frac{\pi}{3} - \sin 0 - i\left(\cos\frac{\pi}{3} - \cos(0)\right)\right]$$

$$= \frac{1}{2}\left[\frac{\sqrt{3}}{2} - 0 - i\left(\frac{1}{2} - 1\right)\right] = \frac{1}{2}\left[\frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right)\right]$$

$$= \frac{\sqrt{3}}{4} + i\left(\frac{1}{4}\right)$$

Properties of definite Integral

(i) $Re\left[\int_a^b w(t)\,dt\right] = \int_a^b [Re(w(t))]\,dt$

(ii) $Im\left[\int_a^b w(t)\,dt\right] = \int_a^b [Im\, w(t)]\,dt$

(iii) $\int_a^b [w(t) + z(t)] = \int_a^b w(t)\,dt + \int_a^b z(t)\,dt$

(iv) $\int_a^b z_0\, w(t)\,dt = z_0 \int_a^b w(t)\,dt$, where z_0 is constant.

LEMMA :

If $w(t)$ is continuous Complex valued functions defined on the interval (a, b) then

$$\left|\int_a^b w(t)\,dt\right| \leq \int_a^b |w(t)|\,dt$$

Proof:

Let $\int_a^b w(t)\,dt = r_0 e^{i\theta_0}$ where r_0 is the moduls function and θ_0 is the arg function. We have $r_0 = e^{-i\theta_0}\int_a^b w(t)\,dt$ — ⊛

using 4^{th} Property we have

$r_0 = \int_a^b e^{-i\theta_0} w(t)\,dt$ and we known that r_0 is always real.

$\therefore r_0 = R.P \int_a^b e^{-i\theta_0} w(t)\,dt$, using 1^{st} Property we have,

$$r_0 = \int_a^b \left[R.P\; e^{-i\theta_0} w(t)\,dt\right] \quad \text{— (1)}$$

and we known that

$Re(z) \leq |z| \Rightarrow$

$$Re\left(e^{-i\theta_0} w(t)\right) \leq \left|e^{-i\theta_0} w(t)\right| \quad \text{— (2)}$$

Subsitituting ② in ① ⇒

we have

$$r_0 \le \int_a^b \left| e^{-i\theta_0} w(t) \right| dt$$

Schwartz formula $[\because |a\,b| \le |a||b|]$

$$r_0 \le \int_a^b \left|e^{-i\theta_0}\right| |w(t)| dt$$

$$\le \int_a^b 1 . |w(t)| dt$$

$\because \left|e^{-i\theta}\right| = 1$

$$r_0 \le \int_a^b |w(t)| dt \quad \text{③}$$

By ⊛ in ③

$$\Rightarrow \left| e^{-i\theta_0} \int_a^b w(t)\, dt \right| \le \int_a^b |w(t)| dt$$

$$\left|e^{-i\theta_0}\right| \left| \int_a^b w(t) dt \right| \le \int_a^b |w(t)| dt$$

$$\left| \int_a^b w(t) dt \right| \le \int_a^b |w(t)| dt$$

Differentiable curve:

A curve c is said to be a diff. if $z'(t)$ exists and is continuous. If $z'(t)$ is $\neq 0$ then the curve is said to be regular smooth curve (or) non-singular.

Piece wise differentiable

A curve c given by $z = z(t)$ is said to be piecewise differentiable. if its diff. except at a finite number of points.

Simple arc (or) Jordan arc (not closed) :

Suppose $z(t)$ is a continuous complex function defined on the $[a,b]$ then the set of points $z(t)$ where $a \le t \le b$ is called a Jordan arc if $t_1 \neq t_2 \Rightarrow z(t_1) \neq z(t_2)$

Simple closed (or) Jordan curve.

The simple arc is called a simple closed curve (or) Jordan closed curve, If $z(a) = z(b)$ and for any two distinct pts. $z(t_1) \neq z(t_2)$

Line Integral

Let C be piecewise diff. curve given by $z = z(t)$, where $a \le t \le b$. Let $f(z)$ be a continuous complex valued fn defined in a region contain in the curve C. Then the line integral of a $f(z)$ is $\int_C f(z)\,dz$ is defined by

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$

Contour

A Contour (or) piecewise smooth arc is one consisting of a finite number smooth arc joined end to end.

Eg. Polygonal path is a Contour.

Problems.

1. Consider $\int_C f(z)\,dz$ where $f(z) = \frac{1}{z}$ and c is the circle $|z| = r$ discribe in positive sense (clockwise)

Soln:

Given $f(z) = \frac{1}{z}$, $|z| = r$. The region is $-r \le t \le r$ and w.k.T $z(t) = re^{it} \Rightarrow z'(t) = rie^{it}$

$$f(z) = \frac{1}{z} = \frac{1}{re^{it}}$$

$$\int_C f(z)\,dt = \int_C f(z(t))\,z'(t)\,dt = \int_{-r}^{r} \frac{1}{re^{it}}\left(rie^{it}\right)dt$$

$$= i \int_{-r}^{r} (1)\, dt$$

$$\therefore \int_c f(z)dt \quad = i\, [t]_{-r}^{r} = 2ir$$

(x) 2). Prove that $\int_{-c} f(z)dz = -\int_c f(z)dz$.

Solution:

The contour $-c$ represent the parametric curve $z(t) = z(-t)$

$$\int_{-c} f(z)dz = -\int_c f(z)dz \quad \text{—} ①$$

$$\text{L.H.S} \quad \int_{-c} f(z)dz = \int_{-b}^{-a} f(z(-t))\, z'(t)\, d(-t)$$

where, considering the $z(-t)$ is derivative of $z(t)$ is evaluated at $-t$.

Let $-t = u \Rightarrow -dt = du$

If $z(-t) = z(t)$

$$z'(-t)d(-t) = z'(u)du$$

$$-z'(-t)dt = z'(u)du$$

$$t = -t \;,\; u = -(-t) \;,\; u = b$$

$$\therefore \text{L.H.S} = \int_b^a f(z(t))\, z'(u)\, du$$

$$= -\int_a^b f(z(t))\, z'(u)du$$

$$= -\int_c f(z)dz \;//$$

Note:

Suppose C Consistist of two Converse C_1 & C_2 from z_1 to z_2 followed by z_2 to z_3, where the initial poind of C_2 co-inside with terminal point of C_1. (ie) $C = C_1 + C_2$

(ie) $\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$

Here C is called sum of its logs to c_1 and c_2. Illy the Contours $C_1, -C_2$ are also well defined when they have same terminal points and stated as $C = C_1 - C_2$.

Result:

⊗ Lemma:

$$\left| f(z)dz \right| \leq ML,$$ where M is any non-negative constant such that $|f(z)| \leq M$ and L denotes length of the contour set. (or) maximum modulus Principle.

Proof:

By defn. of line Integral

$$\int_C f(z)dz = \int f(z(t))\, z'(t)\, dt$$

Given $|f(z)dz| \leq ML$

$$|f(z)| \leq M$$

$$|f(z)dz| = \left| \int f(z(t))\, z'(t)\, dt \right|$$

$$\leq \int_C |f(z(t))|\, |z'(t)|\, dt$$

$$\leq \int M\, |z'(t)|\, dt$$

$$\leq M \int |z'(t)|\, dt \qquad [\because L = \int z'(t)\, dt$$

$$\leq ML$$

" Moduls of the Value of the integral of f along c does not exceed ML.

~~Eg (1)~~ ~~Evaluate~~
Ex: (1) Evaluate $I_1 = \int_c z^2 dz$ where C_1 is the line segment $z=0$ to $2+i$.

Solution:

Given $I_1 = \int_{c_1} z^2 dz$, $z=0$ to $2+i$

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\frac{y-0}{1-0} = \frac{x-0}{2-0} \Rightarrow \frac{y}{1} = \frac{x}{2}$$

$$\boxed{2y = x}$$

Let $z = x+iy$

$= 2y+iy = y(2+i)$

$z'(t) = (2+i)$

$f(z(t)) = (y(2+i))^2$

$$\int_c f(z)dz = \int_c f(z(t))\, z'(t)dt$$

$$= \int_0^{2+i} (y(2+i))^2 (2+i)dy$$

$$= (2+i)^3 \int_0^{2+i} y^2 dy = (2+i)^3 \left[\frac{y^3}{-3}\right]_0^{2+i}$$

$$= \frac{(2+i)^6}{3}$$

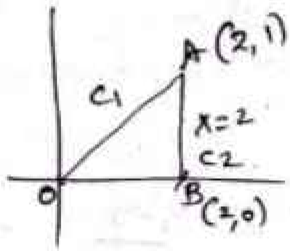
(or) $= (2+i)^3 \left.\frac{y^3}{3}\right]_0^1 = \frac{(2+i)^3}{3}$

HW. (2). Evaluate function $f(z) = y - x - i3x^2$, $z = 0$ to $1+i$

Ans: $-9/2$

3. Evaluate.

$$I_2 = \int_{C_2} z^2 dz$$ where C_2 is denoted by the Contour OBA.



Solution:

where $C_2 = OB + BA$

Along OB, $\underset{x_1, y_1}{(0,0)}$ $\underset{x_2, y_2}{(2,0)}$

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\Rightarrow \frac{y - 0}{0 - 0} = \frac{x - 0}{2 - 0} \Rightarrow 2y = 0$$

$$\boxed{y = 0}$$

$\Rightarrow$ It is in x-axis

(ie) $z = x \Rightarrow dz = dx$

$$\int_{OB} f(z)\,dz = \int_0^2 f(z(t))\,z'(t)\,dt$$

$$= \int_0^2 x^2\,dx \Rightarrow \frac{x^3}{3}\Big]_0^2 = 8/3$$

Along BA, (2,0) (2,1)

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\frac{y - 0}{1 - 0} = \frac{x - 2}{2 - 2}$$

$$\frac{y}{1} = \frac{x - 2}{0}$$

$$x - 2 = 0$$

$$\boxed{x = 2}$$

$z = x+iy$

$= 2+iy$

$dz = i\,dy$

$$\int_{BA} f(z)\,dz = \int_0^1 (2+iy)^2\, i\,dy$$

Put $t = 2+iy$

$dt = i\,dy$

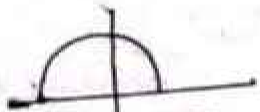
$dy = -dt$

$$= \int_0^1 t^2\,dt = \left.\frac{t^3}{3}\right|_0^1 = \frac{(2+i)^3}{3} - \frac{8}{3} = \frac{11i-6}{3}$$

$$\therefore \int_C z^2\,dz = \int_{OB} f(z)\,dz + \int_{BA} f(z)\,dz$$

$$= \frac{8}{3} + \frac{11i}{3} - \frac{6}{3} = \frac{11i+2}{3} \;//$$

4. Evaluate: $I_3 = \int_{C_3} \bar{z}\,dz$, where C_3 is upper half of the Circle $|z|=1$.



$\theta = 0$ to π

solution: Let $z = re^{i\theta}$, $r=1$

$\therefore$ $z = re^{i\theta}$ is the Complex Valued function.

$\bar{z} = \overline{(e^{i\theta})} = e^{-i\theta} = f(z(t))$

$z(t) = e^{i\theta}$

$dz = ie^{i\theta}d\theta$

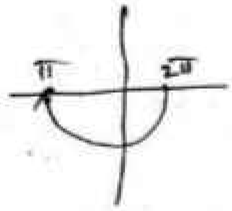
$$\int_{C_3} f(z)\,dz = \int_0^{\pi} f(z(t))\,z'(t)\,dt$$

$$= \int_0^{\pi} e^{-i\theta}\cdot ie^{i\theta}d\theta$$

$$= \int_0^{\pi} i\,d\theta$$

$= i\pi \;//$

5. Evaluate: $I_H = \int_{C_H} \bar{z}\, dz$ where C_H is lower of half the circle $|z|=1$



Solution:

Let $z = re^{i\theta}$, $r=1$

$\therefore z = re^{i\theta}$ is the Complex valued functions.

Now $\bar{z} = \overline{(e^{i\theta})}$, $f(z(t)) = e^{-i\theta}$

$z(t) = e^{i\theta}$

$dz = i\, e^{i\theta} d\theta$

$$\int_{C_H} f(z)\,dz = \int_{\pi}^{2\pi} e^{-i\theta} . i e^{i\theta} d\theta$$

$$= \int_{\pi}^{2\pi} i\, d\theta = i \int_{\pi}^{2\pi} d\theta = i\, [\theta]_{\pi}^{2\pi} = i\pi_{//}$$

6. Evaluate: I_0 of $\bar{z}$ taken around the circle $C_0 = C_3 + C_H$ in anticlock wise direction of the circle $|z|=1$.

Solution:

Given $C_0 = C_3 + C_H$

$= i\pi + i\pi$

$C_0 = 2i\pi$

Cauchy Theorem:

Statement:

Let the functions $f(z)$ be analytic in a domain D and $f'(z)$ be Continuous in D then $\int_C f(z)dz = 0$ for every simple closed contour C in D.

Proof: Let $z = z(t)$, $a \le t \le b$ represent a simple closed contour c. then by formula

$$\int_c f(z)dz = \int_c f(z(t))\, z'(t)\,dt$$

If $f(z) = u(x,y) + iv(x,y)$

$z(t) = x(t) + iy(t)$

then $f(z(t))\, z'(t)$ is the product of functions.

$$u[x(t), y(t)] + iv[x(t), y(t)] \text{ and } x'(t) + iy'(t)$$

$$\int_C f(z)dz = \int_a^b [u'_x + iu'_y + iv'_x - v'_y]dt$$

$$= \int_a^b (u'_x - v'_y)dt + i\int_a^b (v'_x + u'_y)dt$$

$$= \int_C (U + iV)(dx + idy)$$

$$= \int_C (u\,dx - V\,dy) + i\int_C V\,dx + U\,dy \quad \text{——} (1)$$

$\because f'$ is continuous in D, first derivatives of U and V are continuous.

By Green's theorem.

$$\int_C P\,dx + \int_C Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

(1) R.P

$$\int_C U\,dx + \int_C -V\,dy = \iint_R \left(-\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right) dx\,dy \quad \text{——} (2)$$

(1) – I.P

$$\int_C v\,dx + \int U\,dy = \iint_R \left(\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y}\right) dx\,dy \quad \text{——} (3)$$

Sub (2) & (3) in (1)

$$\Rightarrow \int_C f(z)\,dz = \iint_R [-V_x - U_y + iU_x - iV_y]\, dx\,dy$$

C.R eqns. $U_x = V_y$

$U_y = -V_x$

$$\Rightarrow \int_C f(z)dz = 0,$$

2m

Cauchy Goursat's theorem

Statement:

If functions $f(z)$ is analytic at all points inside and on C, then $\int f(z)\,dz = 0$.

Simply Connected domain:

A simply connected domain, if $\exists$ every simple closed contour with in n closes only point of D.

Note:

A domain which is not simply connected is multiply connected.

Winding Number:

Let c be a closed contour and z_0 be a pt. not on c, then

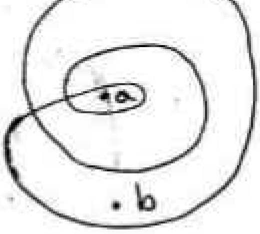
$$\eta(c, z_0) = \frac{1}{2\pi i}\int_c \frac{1}{z - z_0}\,dz$$ is called Winding number of c, around z_0 (or) Index of C with respect to z_0.

Eg. 1

$\eta(c, a) = 1$

$\eta(c, b) = 0$

Eg. 2



$\eta(c, a) = 3$

$\eta(c, b) = 1$

Properties of Winding Number:

If c is a closed contour and z_0 is a pt. not on c, then $\eta(c, z_0) = \eta(c_1, z_0) + \eta(c_2, z_0)$

where $c = c_1 + c_2$

* $\eta(-c, z_0) = -\eta(c, z_0)$.

Theorem: Cauchy Integral formula:

Statement: Let the function $f(z)$ be a analytic with in and on a simple closed Contour c, taken in the Positive sense and z_0 be any pt. interior to C.

Then $f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0} dz$.

Proof: Let C_r denote the circle $|z-z_0|=r$ where, the radius are is so, small that C-R lies entirely inside c,

Consider the functions $\frac{f(z)}{z-z_0}$ which is analytic with in and on the Contour C_r and C.

$$\therefore \int_C \frac{f(z)}{z-z_0} dz = \int_{C_r} \frac{f(z)}{z-z_0} dz$$

$$= \int_{C_r} \frac{f(z_0)+f(z)-f(z_0)}{z-z_0} dz$$

$$= \int_{C_r} \frac{f(z_0)}{z-z_0} dz + \int_{C_r} \frac{f(z)-f(z_0)}{z-z_0} dz$$

$$= \int_{C_r} \frac{f(z)-f(z_0)}{z-z_0} dz + 2\pi i f(z_0)$$

Let

$$z-z_0 = re^{i\theta}, \quad 0 \le \theta \le 2\pi$$

$$dz = re^{i\theta} i\, d\theta$$

$$\int \frac{dz}{z-z_0} = \int_{C_r} \frac{re^{i\theta} i\, d\theta}{re^{i\theta}} = i \int_{C_r} d\theta$$

$$= i \int_0^{2\pi} d\theta = i\theta\Big]_0^{2\pi} = 2\pi i$$

Sub. $$\int_C \frac{f(z)}{z-z_0} dz = \int_{C_r} \frac{f(z)-f(z_0)}{z-z_0} dz + f(z_0)\, 2\pi i$$

$\div\, 2\pi i$

$$\Rightarrow \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}\,dz + f(z_0)$$

As $f(z_0)$ is Continuous and z given $\epsilon_1 > 0$ $\exists$ $\delta > 0 \ni$

$$|f(z)-f(z_0)| < \epsilon_1 \text{ , whenever } |z-z_0| < \delta$$

$$\therefore \left|\frac{1}{2\pi i}\int_C \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| < \frac{1}{2\pi}\,\frac{\epsilon}{P}\,2\pi P = \epsilon$$

$$\Rightarrow \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz = \epsilon + f(z_0)$$

Since ϵ_1 is arbitraly, we can take only the function of an avalticity.

$$\therefore \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z0}\,dz = f(z_0)$$

(or)

$$\therefore \int_C \frac{f(z)}{z-z0}\,dz = 2\pi i\, f(z_0)$$

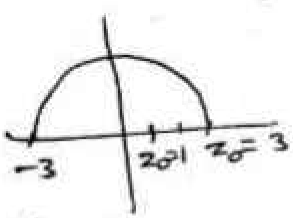
Problems:

(1). Evaluate: $\displaystyle\int_C \frac{e^{2z}\,dz}{(z-1)(z-2)}$, c is the circle $|z|=3$.

Solution:

Given $|z|=3$ and $f(z)=e^{2z}$ is analytic with in the circle $|z|=3$, whose singular pt. of given by

$$z_0-1=0 \quad \Big| \quad z_0-2=0$$
$$z_0=1 \quad \Big| \quad z_0=2$$



$\therefore$ The pts. $z_0=1$, $z_0=2$ lies inside the circle.

$\therefore$ By. Cauchy formula

$$\int_C \frac{f(z)}{z-z0}\,dz = 2\pi i\, f(z_0)$$

Let $f(z) = e^{2z}$

$f(z_0) = e^{2z_0}$

$$\int_c \frac{e^{2z}\,dz}{(z-1)(z-2)} = \int_c e^{2z}\left[\frac{1}{z-2} - \frac{1}{z-1}\right]dz \qquad \because f(z_0) = e^{2z_0}$$

$f(1) = e^{2z}$

$f(2) = e^{4z}$

$$= \int_c \frac{e^{2z}}{(z-2)}dz - \int \frac{e^{2z}\,dz}{(z-1)}$$

$$= 2\pi i(e^4) - 2\pi i(e^2)$$

$$= e^2 2\pi i[e^2-1]_{11}$$

2. Evaluate: $\int_c \frac{\cosh \pi z}{z(z^2+1)}dz$

sol:

$f(z) = \cosh\pi z$ which is analytic singular pt. of given by

$z=0 \qquad z^2+1=0$

$z^2=-1$

$z = \pm 1$ and $z = i, -i$

$\therefore$ The pts. $z=0, z=i, z=-i$

Consider $\int_c \frac{\cosh\pi z}{z(z^2+1)}$, then $f(z) = \cos i\pi z$

$$\frac{\cosh\pi z}{z(z^2+1)} = \frac{A}{z} + \frac{B}{z+i} + \frac{C}{z-i}$$

$$\cosh\pi z = A(z-i)(z+i) + B(z-i)z + C(z+i)z$$

Put $z=0$

$\cos i\pi(0) = A(-i)(i)$

$1 = A$

Put $z = i$

$$\cos i\pi(i) = A(0) + B(0) + C(2i)i$$

$$\cos(-\pi) = 2C(-1)$$

$$1 = -2C$$

$$\boxed{C = -\frac{1}{2}}$$

Put $z = -i$

$$\cos i\,\pi(-i) = B(-2i)(-i)$$

$$\cos\pi\,(-1) = -2B$$

$$1 = -2B$$

$$\boxed{B = -\frac{1}{2}}$$

$$\frac{\cosh\pi z}{z(z^2+1)} = \left[\frac{1}{z} - \frac{1}{2(z+i)} - \frac{1}{2(z-i)}\right]$$

$$\int_C \frac{\cosh\pi z}{z(z^2+1)}\,dz = \int_C \frac{1}{z} - \frac{1}{2(z+i)} - \frac{1}{2(z-i)}(\cosh\pi z)\,dz$$

$$= \int \frac{\cosh\pi z}{z}\,dz - \frac{1}{2}\int_C \frac{\cosh\pi z}{(z+i)}\,dz - \frac{1}{2}\int_C \frac{\cosh\pi z}{z-i}\,dz$$

$$= 2\pi i f(0) - \frac{1}{2}2\pi i\, f(-i) - \frac{1}{2}2\pi i f(i)$$

$$f(z) = \cos i\pi z \Rightarrow f(0) = \cos i\pi(0) = 1$$

$$f(-i) = \cos i\pi(-i) \Rightarrow f(-i) = \cos\pi(-1) = -1$$

$$f(i) = \cos i\pi(i) \Rightarrow f(i) = \cos\pi(-1) = -1 \qquad \cos(-\pi)$$

$$\Rightarrow 2\pi i(1) + \frac{1}{2}2\pi i(1) - \frac{1}{2}2\pi i(1)$$

$$\Rightarrow 2\pi i + \pi i + \pi i$$

$$\Rightarrow 2\pi i + 2\pi i = 4\pi i$$

$$\int_C \frac{\cosh\pi z}{z(z^2+1)} = 4\pi i$$

(4) Derivatives of analytic function:

Statement:

Let $f(z)$ be a analytic within and on a simple closed contour C taken in +ve sense then for any pt. z_0 interior to C, then

$$f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^2}\, dz$$

Proof:

Take a pt. z_0 interior to C and choose $\Delta z_0 \ni z_0 + \Delta z_0$ is also interior to C.

By Cauchy Integral formula,

$$\therefore f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0}\, dz \quad \text{——} ①$$

Let d be the smallest distance from z_0 to the pt. z on C.

$$\therefore \frac{f(z+\Delta z_0) - f(z_0)}{\Delta z_0} = \frac{1}{\Delta z_0 2\pi i} \int_C \left[\frac{f(z)}{z-z_0-\Delta z_0} - \frac{f(z)}{z-z_0} \right] dz$$

$0 < |z| < d$,
$$= \frac{1}{\Delta z_0 2\pi i} \int_C f(z) \left[\frac{z-z_0-z+z_0+\Delta z_0}{(z-z_0)(z-z_0-\Delta z_0)} \right] dz$$

$$= \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{(z-z_0)(z-z_0-\Delta z_0)}$$

Adding & Subtracting $\frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^2} dz$

$$\frac{f(z_0+\Delta z_0) - f(z_0)}{\Delta z_0} - \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^2} dz = \frac{1}{2\pi i} \int_C \left[\frac{1}{(z-z_0-\Delta z_0)(z-z_0)} - \frac{1}{(z-z_0)^2} \right] f(z)\, dz$$

$$= \frac{1}{2\pi i} \int_C \left[\frac{(z-z_0) - (z-z_0-\Delta z_0)}{(z-z_0-\Delta z_0)(z-z_0)^2} \right] f(z)\, dz$$

$$= \frac{1}{2\pi i} \int_C \left[\frac{z-z_0-z+z_0+\Delta z_0}{(z-z_0)^2 (z-z_0-\Delta z_0)} \right] f(z)\, dz$$

$$= \frac{1}{2\pi i} \int_C \left[\frac{\Delta z_0}{(z-z_0)^2 (z-z_0-\Delta z_0)} \right] f(z)\, dz \qquad \text{— (2)}$$

$$\Rightarrow \left| \frac{f(z_0+\Delta z_0) - f(z_0)}{z_0} - \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^2} dz \right| = \left| \frac{1}{2\pi i} \int_C \frac{\Delta z_0 f(z)}{(z-z_0-\Delta z_0)(z-z_0)^2} dz \right|$$

Let M be maximum value of $|f(z)|$ on C. Since $|z-z_0| \geq d$ and $|\Delta z_0| < d$

$$\therefore \quad |z-z_0-\Delta z_0| \geq \big| |z-z_0| - |\Delta z_0| \big|$$
$$\geq d - |\Delta z_0|$$
$$> 0$$

$$\therefore \left| \int_C \frac{\Delta z_0 f(z)}{(z-z_0-\Delta z_0)(z-z_0)^2} dz \right| \leq \frac{|z_0| ML}{(d-|\Delta z_0|)d^2}$$

where L is length of C.

Let $\Delta z_0 \to c$ then according to the above inequality R.H.S of (2) also tends to 0

$$\lim_{\Delta z_0 \to 0} \frac{f(z_0+\Delta z_0) - f(z_0)}{\Delta z_0} - \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^2} dz = 0$$

$$\Rightarrow \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z_0) - f(z_0)}{\Delta z_0} = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^2} dz$$

$$\Rightarrow f'(z) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^2} dz$$

Problems:

1. Evaluate $\frac{1}{2\pi i} \int_C \frac{z^2+5}{z-3} dz$, C is $|z|=4$

Soln:

Given $|z|=4$ and $f(z) = z^2+5$ is analytic with in the circle $|z|=4$ whose by singular pt. of given by $z_0 = 3$

$$f(z_0) = \frac{1}{2\pi i} \int \frac{f(z)}{z-z_0} dz$$

$$f(3) = \frac{1}{2\pi i} \int_C \frac{z^2+5}{z-3} dz$$

$$f(3) = f(z^2+5)$$

$$= z^2+5 = 14$$

2. Evaluate, Using $\int_C \frac{z}{z-2} dz$, C is $|z|=1$

solution:

Given $|z|=1$

$f(z) = z$ is analytic with lies out side C given by

$z-2=0$

$\Rightarrow z=2$

$f(z)$ is not always analytic outside

$\therefore \int_C f(z) dz = 0$.

3. Evaluate $\int_C \frac{\sin 3z}{z+\pi/2}$, C is $|z|=5$

Given $|z|=5$, $f(z)=\sin 3z$ is analytic with in the circle $|z|=5$ whose singular point is given by $z_0=-\frac{\pi}{2}$

$$\int_C \frac{f(z)}{z-z_0}\,dz = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz$$

$$= 2\pi i \sin f(z_0)$$

$$= -2\pi i\left(\sin \frac{3\pi}{2}\right) = -2\pi i(-1) = 2\pi i$$

4. Evaluate $\int_C \frac{3z-1}{z^3-z}\,dz$, C is $|z|=\frac{1}{2}$

solution: Given $|z|=\frac{1}{2}$, $f(z)=3z-1$ is analytic with in the circle $|z|=\frac{1}{2}$ whose singular pt. are

$$\frac{3z-1}{z^3-z} = \frac{3z-1}{z(z^2-1)}$$

$$\frac{3z-1}{z(z^2-1)} = \frac{A}{z}+\frac{B}{z+1}+\frac{C}{z-1}$$

$$3z-1 = A(z+1)(z-1)+Bz(z-1)+C(z+1)z$$

Put $z=0$	Put $z=-1$	Put $z=1$
$-1=A(1)(-1)$	$-4=B(-1)(-2)$	$2=C(2)1$
$-1=-A$	$-4=B(2)$	$2=C(2)$
$\boxed{A=1}$	$\boxed{B=-2}$	$\boxed{C=1}$

$$\int_C \frac{3z-1}{z(z^2-1)} = \int_C \frac{3z-1}{z}\,dz - 2\int_C \frac{3z-1}{z+1}\,dz + \int_C \frac{3z-1}{(z-1)}\,dz$$

$$= 2\pi i f(0) - 2(2\pi i) f(-1) + 2\pi i f(1)$$

$$= 2\pi i\,(3(0)-1) - 4\pi i(3(-1)-1) + 2\pi i(3(1)-1)$$

$$= -2\pi i + 8\pi i + 4\pi i$$

$$= 10\pi i$$

Seen,
26/7/2024